

Einstein-Yang-Mills wormholes haunted by a phantom field

(Including the errata, last updated on November 27, 2025)

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Errata

As with most mathematical texts, some errors managed to slip through the proofreading process. I list the errors here rather than directly update the text in order to keep the original dissertation intact. One may notice that most of these errors occurred in §3, which in hindsight makes sense as this chapter was written rather hastily directly prior to submitting.

- ◇ In §3.1.1, p. 22, first item on the top: The exterior derivative is not well-defined in this way. Here, one should assume that the bundle E is a trivial vector bundle $E = M \times V$ for some vector space V , and the formula should be

$$d(\theta \otimes \sigma) = (d\theta) \otimes \sigma + (-1)^{\deg \theta} \theta \wedge (d\sigma).$$

This does not affect any of the later definitions and calculations. The error went unnoticed because the exterior derivative on bundle-valued forms is later applied only in the situation as described above, where in addition σ remains a constant element of V .

- ◇ In §3.1.1, p. 22., second item from the top: The pullback should also appear on the second factor

$$\phi^*(\theta \otimes \sigma) = (\phi^*\theta) \otimes (\phi^*\sigma),$$

and the pullback bundle should be

$$\phi^*\Omega^k(N, E) = \Gamma(\Lambda^k(\phi^*T^*N) \otimes \phi^*E), \quad \phi^*E = \coprod_{x \in M} E_{\phi(x)}.$$

- ◇ In Theorem 3.2.3 (i), p. 30, for the classification to be complete one should additionally assume that any principal Z -bundle over N is trivial, where Z denotes the center of $\lambda(H) \subset G$. This is achieved e.g. if N is contractible. In the context of the thesis, we have $N = \mathbb{R}^2$ which is contractible, so the classification is complete.
- ◇ In Theorem 3.2.3 (ii), p. 30, in the second point η should have not have values in $\mathfrak{h}$, but rather in $\mathfrak{g}$, in particular the subalgebra of $\mathfrak{g}$ that is invariant under the adjoint action of $\lambda(H)$. Correspondingly, in the first paragraph on p. 33, the form should have values in the subalgebra of $\mathfrak{su}(2)$ that is invariant under the adjoint action of $\mathbf{U}(1)$. The conclusions, in particular Theorem 3.2.5, are correctly stated.
- ◇ In the Bibliography: Reference [HSV08] is incorrectly dated - the year should be 1980 instead of 2008.

If you find any additional errors while reading, please do not hesitate to contact me at:

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Abstract

In this thesis, we study wormhole spacetimes in the framework of the static spherically symmetric $\mathbf{SU}(2)$ Einstein-Yang-Mills theory coupled to a phantom scalar field. After presenting all the necessary background, we show rigorously the existence of an infinite sequence of symmetric wormhole solutions, labelled by the number of zeros of the Yang-Mills potential. These solutions have previously been discovered numerically. Mathematically, the problem resembles the pure Einstein-Yang-Mills system for black hole initial conditions, which was well-studied in the 90s. The main difference in the present work is that the coupling to the phantom field adds a non-trivial degree of complexity to the analysis. After proving the existence of the symmetric wormhole solutions, we also present numerical evidence for the existence of asymmetric ones.

Zusammenfassung

In dieser Arbeit werden WurmLöcher im Rahmen der statischen sphärisch-symmetrischen $\mathbf{SU}(2)$ Einstein-Yang-Mills Theorie, zusätzlich gekoppelt mit einem Phantom-Skalarfeld, untersucht. Nachdem alle relevanten Hintergrundinformationen erläutert wurden, zeigen wir die Existenz einer unendlichen Folge von symmetrischen Wurmlochlösungen, die durch die Anzahl der Nullstellen des Yang-Mills Potenzials indiziert ist. Diese Lösungen wurden zuvor numerisch entdeckt. Mathematisch betrachtet, hat das Problem Gemeinsamkeiten mit dem Einstein-Yang-Mills-System mit den Anfangsbedingungen von schwarzen Löchern, welches bereits in der Vergangenheit umfassend analysiert wurde. Der Hauptunterschied besteht darin, dass das Phantom-Skalarfeld einen nicht-trivialen Grad an Komplexität in die Analyse hinzufügt. Nachdem die Existenz der symmetrischen Wurmlochlösungen bewiesen wurde, präsentieren wir außerdem die Ergebnisse unserer numerischen Analyse, welche die Existenz von asymmetrischen Wurmlochlösungen zeigen.

Acknowledgements

The author would like to express his gratitude to Anna Siffert, for suggesting the problem studied in this thesis, as well as for the helpful discussions surrounding it. Many thanks also to Gustav Holzegel for suggesting certain improvements to the text.

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Chapter 1

Introduction

1.1 Wormholes

Wormholes are hypothetical models of spacetime that describe traversable tunnels connecting asymptotically flat universes. Despite their ubiquity in popular culture, wormholes are indeed still hypothetical from a physics standpoint because they require support from so-called exotic matter [MT88; Vis96; Hay99]. From a mathematical perspective, possibly the most natural way of modelling such matter is by means of a phantom field (or ghost), which is a scalar field with a reversed sign in front of its energy density term in the Lagrangian. Such fields often appear in cosmological research, as they could explain the accelerated expansion of the universe [Cal02].

The first examples of wormholes supported (or, somewhat more poetically, “haunted”) by phantom fields were constructed independently by Ellis [Ell73] and Bronnikov [Bro73]. Their model is relatively simple, so let us discuss it in more mathematical detail. Intuitively, the first geometric shape that might come to mind when imagining a wormhole is a catenoid, cf. Figure 1.1. This surface can be viewed as a surface of revolution in $\mathbb{R}^3$ parametrized by

$$(\rho, \varphi) \mapsto (\cosh \rho \cos \varphi, \cosh \rho \sin \varphi, \rho), \quad (1.1)$$

with induced metric

$$\cosh^2(\rho) (d\rho \otimes d\rho + d\varphi \otimes d\varphi).$$

Aiming to extend this metric to a spacetime metric, we further introduce a trivial time coordinate t and extend φ to a spherical angle pair $(\theta, \varphi) \in \mathbb{S}^2$, thus obtaining the metric

$$g = -dt \otimes dt + \cosh^2(\rho) (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

where $g_{\mathbb{S}^2}$ is the round metric on $\mathbb{S}^2$ given by

$$g_{\mathbb{S}^2} = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi.$$

The Lorentzian metric g is globally regular and has two asymptotically flat ends as $\rho \rightarrow \pm\infty$. A simple calculation yields

$$\text{Ric}_g = -2 \text{sech}^2(\rho) d\rho \otimes d\rho,$$

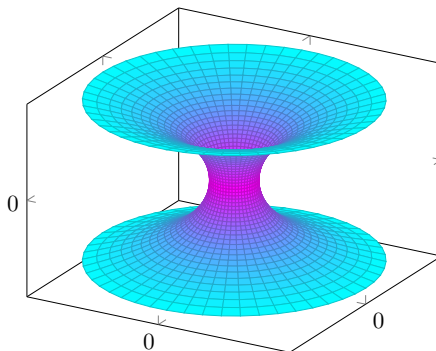


Figure 1.1: Embedding diagram of the Ellis-Bronnikov wormhole.

which shows that such a metric does not satisfy the Einstein field equations in vacuum $\text{Ric}_g = 0$. This is unsurprising since the celebrated theorem of Birkhoff [BL23] ensures that the only spherically symmetric solution of the vacuum Einstein equations is the Schwarzschild solution. However, the crucial observation of Ellis and Bronnikov was that the metric g can be viewed as the solution of the haunted Einstein field equations

$$\begin{cases} \text{Ric}_g + d\phi \otimes d\phi = 0, \\ \square_g \phi = 0, \end{cases}$$

where $\square_g$ is the wave operator with respect to g and ϕ is the scalar phantom field

$$\phi(\rho) = \sqrt{2} \arctan(\sinh \rho) = \sqrt{2} \text{gd}(\rho),$$

where gd is the so-called *Gudermannian function* [Zwi03]. This theory necessitates that the scalar field term has an “incorrect” sign in the field equations, which can be interpreted as ϕ being a negative energy field. Nevertheless, the metric is now a well-defined solution of a relativistic theory. The spacetime $\mathbb{R}^2 \times \mathbb{S}^2$ equipped with the metric g is universally known as the *Ellis-Bronnikov wormhole*. It has two asymptotically flat ends at $\rho = \pm\infty$, connected by a tunnel which is at its *throat* (i.e. at $\rho = 0$) foliated by spheres of minimal radius. This is quite nicely seen in the so-called *embedding diagram* of the wormhole in Figure 1.1, which is in the case of the Ellis-Bronnikov wormhole simply the catenoid parametrized by the map (1.1).

The Ellis-Bronnikov wormhole is a prototypical example of a *static spherically symmetric wormhole* (sometimes also known as a *Morris-Thorne wormhole*). They were mathematically defined in the pioneering work [MT88] of Morris and Thorne, and subsequently studied by many other authors, see e.g. the book [Vis96] of Visser. These wormholes are modelled as spacetimes of the form $M = \mathbb{R}^2 \times \mathbb{S}^2$ equipped with the Lorentzian metric

$$g = -e^{2\tau(\rho)} dt \otimes dt + r(\rho)^2 (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

where τ, r are smooth functions with $r > 0$, and such that the spacetime is asymptotically flat as $\rho \rightarrow \pm\infty$. The metric g should additionally respect the theory of general relativity,

in the sense that it satisfies the Einstein field equations for some accepted matter model. Wormholes of this type will be the main object of study in this work.

1.2 Einstein-Yang-Mills theory

Yang-Mills theory is one of the many fruits of the 20th century boom in physics. It provides an elegant mathematical framework for the current Standard Model of particle physics, but it is also mathematically interesting in its own right, as demonstrated by the fact that the Yang-Mills existence and mass gap problem is one of the (still unsolved) Millenium Prize Problems. In mathematical terms, Yang-Mills (YM) theory revolves around studying connections on principal G -bundles for some compact Lie group G (called the *gauge group*), which are critical points of the YM functional

$$\omega \mapsto \int \|F_\omega\|^2 \text{vol}_g.$$

Here, (M, g) is the base semi-Riemannian manifold, ω is a connection on a principal G -bundle over M with curvature F_ω , and the norm $\|\cdot\|$ is induced by both the metric g and a bi-invariant metric on the Lie algebra $\mathfrak{g}$ of G . One may also incorporate gravity into the theory to get the so-called Einstein-Yang-Mills (EYM) theory, which studies critical points of the EYM functional

$$(g, \omega) \mapsto \int (R_g - \|F_\omega\|^2) \text{vol}_g,$$

where R_g is the scalar curvature of g . This theory can be viewed as a generalization of the Einstein-Maxwell theory, since Maxwell's theory of electromagnetism can indeed be viewed a special case of YM theory with gauge group $G = \mathbf{U}(1) \cong \mathbb{S}^1$.

The YM (and consequently also EYM) theory only starts fully shining when the gauge group is chosen to be non-abelian. In fact, the theory is very rich already in one of the simplest cases, namely $G = \mathbf{SU}(2)$. The $\mathbf{SU}(2)$ EYM equations started receiving a lot of attention in the late 20th century, initiated by Bartnik and McKinnon [BM88] when they numerically discovered particle-like globally regular solutions to these equations in the static spherically symmetric setting. Soon after, Bizoń [Biz90] also numerically discovered black hole solutions in this framework. These solutions were rather unexpected [BS93], as they provided some of the first counterexamples to the no-hair conjecture outside the Einstein-Maxwell setting. Indeed, it is known that the Reissner-Nordström family is the unique family of static asymptotically flat black hole solutions to the Einstein-Maxwell system [Heu96], whereas the black holes discovered by Bizoń have non-trivial Yang-Mills “hair”, in the sense that the asymptotic Yang-Mills charges vanish despite the Yang-Mills connection being non-trivial. The existence of these solutions (in fact, infinite sequences thereof) was also shown soon after in a mathematically rigorous manner. This was first done in the series of papers [Smo+91; SW93; SWY93] by Smoller, Wasserman et al. A complete classification of the solutions to the static spherically symmetric $\mathbf{SU}(2)$ EYM equations was later provided in [BFM94] by Breitenlohner, Forgács and Maison, which

also allowed for a somewhat more elegant existence proof. The $\mathbf{SU}(2)$ EYM theory has since been studied in many different aspects, a lot of which have been gathered in the rather extensive review [VG99] of Volkov and Gal'tsov, which the reader can refer to for more details.

1.3 Haunted Einstein-Yang-Mills wormholes

In this work, we study wormholes within the framework of the $\mathbf{SU}(2)$ EYM theory coupled additionally to a phantom field, which we refer to as the *haunted* EYM theory (in contrast, we will refer to the EYM theory with no coupling to a phantom field as the *pure* EYM theory, for clarity). More precisely, we study critical points of the functional

$$(g, \omega, \phi) \mapsto \int (R_g - \|F_\omega\|^2 + \|\mathrm{d}\phi\|^2) \mathrm{vol}_g, \quad (1.2)$$

where ϕ is a scalar function on M . The corresponding Euler-Lagrange equations are referred to as the haunted $\mathbf{SU}(2)$ EYM equations. The main purpose of the manuscript is to mathematically prove the existence of the wormhole solutions to the haunted $\mathbf{SU}(2)$ EYM equations that were discovered numerically in [Hau+14].

Let us briefly present some more mathematical details of the problem. We work in the static spherically symmetric setting, i.e. with spacetimes of the form $M = \mathbb{R}^2 \times \mathbb{S}^2$ equipped with the Lorentzian metric

$$g = -e^{2\tau(\rho)} \mathrm{d}t \otimes \mathrm{d}t + r(\rho)^2 (\mathrm{d}\rho \otimes \mathrm{d}\rho + g_{\mathbb{S}^2}),$$

where τ, r are smooth functions with $r > 0$, and $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$, cf. §1.1. For the $\mathbf{SU}(2)$ Yang-Mills connection, we make the well-known “purely magnetic” static spherically symmetric ansatz

$$\omega = w(\rho) (\mathrm{d}\theta \otimes X + \sin \theta \mathrm{d}\varphi \otimes Y) + \cos \theta \mathrm{d}\varphi \otimes Z,$$

where w is a smooth function, X, Y, Z form the standard basis for $\mathfrak{su}(2)$ induced by the Pauli matrices, and (θ, φ) are the standard coordinates on $\mathbb{S}^2$. Finally, we also assume that the phantom field $\phi = \phi(\rho)$ is static and spherically symmetric. The haunted $\mathbf{SU}(2)$ Einstein-Yang-Mills equations with respect to these ansätze then reduce to the ODE system

$$\left\{ \begin{array}{l} \ddot{w} + \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{w} + w(1 - w^2) = 0, \\ \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{r}\dot{\tau}}{r} - \frac{2\dot{w}^2}{r^2} - \frac{(1 - w^2)^2}{r^2} = 0, \\ \frac{\ddot{r}}{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 + \frac{(1 - w^2)^2}{r^2} = 0, \\ 1 + \frac{2\dot{w}^2}{r^2} - \frac{(1 - w^2)^2}{r^2} - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right) = \frac{\pi_0^2}{(re^\tau)^2}, \end{array} \right. \quad \begin{array}{l} (1.3a) \\ (1.3b) \\ (1.3c) \\ (1.3d) \end{array}$$

where $\pi_0 \in \mathbb{R}$ is a constant that may be chosen freely (in fact, π_0 can be interpreted as the “charge” of the phantom field, since the corresponding term arises from the general solution of the phantom field equation $\square_g \phi = 0$). The final equation (1.3d) is implied by the other three if we assume it holds at least at one point, and therefore we may view it as a constraint on the initial conditions.

Note that putting $\pi_0 = 0$ yields the pure Einstein-Yang-Mills equations. In fact, the main difference in the haunted setting compared to the pure setting is that the introduction of the phantom field allows us to assume that $\tau(0)$ and $\dot{\tau}(0)$ are finite, which is necessary for us since a wormhole spacetime is supposed to be everywhere regular. In contrast, when dealing with initial conditions describing a black hole horizon, one needs to assume that $\tau(0)$ and $\dot{\tau}(0)$ are infinite in an appropriate sense [BFM94, §6], which can also be seen directly from the constraint (1.3d).

Searching for wormhole solutions, we supplement the system (1.3a–1.3d) with the initial conditions

$$\begin{aligned} w(0) &= w_0, & r(0) &= r_0 > 0, & \tau(0) &= \tau_0, \\ \dot{w}(0) &= \dot{w}_0, & \dot{r}(0) &= 0, & \dot{\tau}(0) &= 0. \end{aligned}$$

Here, the choice $\dot{r}(0) = 0$ is quite natural since a wormhole is expected to have asymptotically flat ends at $\pm\infty$, so that $r \rightarrow \infty$ as $\rho \rightarrow \pm\infty$, and in particular there must exist at least one point at which r has a positive minimum. On the other hand, the assumption $\dot{\tau}(0) = 0$ is just made for simplicity (although it is also necessary if one wishes to study symmetric solutions as we will do below). Note that the parameter τ_0 can be kept free because the constant π_0 can be chosen arbitrarily, in particular so that the constraint (1.3d) is satisfied for any (fixed) choice of τ_0 , i.e. we fix

$$\pi_0 = r_0 e^{\tau_0} \sqrt{E_0}, \quad \text{where} \quad E_0 := 1 + \frac{2\dot{w}_0^2 - (1 - w_0^2)^2}{r_0^2} \geq 0.$$

In particular, we will in the end want to fix τ_0 to ensure that we get the correct asymptotic behaviour of τ , thereby eliminating the freedom of this parameter. Thus, we are essentially left with three free parameters $(r_0, w_0, \dot{w}_0)$, which we refer to as the *initial data*. The goal is then to find initial data such that the corresponding solution is globally defined and asymptotically flat at $\pm\infty$, and in particular describes a wormhole spacetime. In fact, in view of the symmetries of the equations, it suffices to study the solutions for $\rho \geq 0$, and then one can deduce their behaviour for $\rho < 0$ by reflection.

The main result can now be roughly stated as follows.

Main theorem. *For each real number $r_0 > 0$ and natural number $n \geq 0$, there exists a haunted $\mathbf{SU}(2)$ Einstein-Yang-Mills wormhole with initial radius r_0 , and such that the Yang-Mills coefficient w has exactly n zeros.*

These wormholes are symmetric, in the sense that the metric coefficients τ and r are even functions, while w is either even or odd with the same parity as n . They also have vanishing Yang-Mills charges at infinity, but non-trivial Yang-Mills potentials, and are therefore also referred to as *hairy wormholes*, in analogy to their hairy black hole counterparts in the pure $\mathbf{SU}(2)$ EYM theory.

To prove the theorem, we follow the blueprint laid forth in the aforementioned work [BFM94] of Breitenlohner, Forgács and Maison. The idea is to use a shooting method argument, i.e. we want to consider infima over certain subsets of initial data, with the expectation that the solutions having these infima as initial data are global and have the desired asymptotic behaviour. To do this in a mathematically satisfactory manner, we essentially need the following three ingredients:

- ◇ the *classification theorem*, which shows that the initial value problem admits only a few possible solution types, depending on whether they are globally defined or not, and depending on the asymptotic values of the dependent variables (note that this also gives a convenient partition of the set of initial data);
- ◇ the *neighbourhood theorem*, which describes how a sufficiently small neighbourhood of each initial data type looks like;
- ◇ the *energy theorem*, which tells us how the solutions with extremal initial energies E_0 (i.e. either very small or very large) behave.

Roughly speaking, the energy theorem provides us with an upper and a lower bound for the shooting method, allowing us to take the aforementioned infima over compact sets. The neighbourhood theorem allows us to deduce which solution types the infima *cannot* correspond to. The exhaustion of all the possible cases from the classification theorem then only leaves one possible option for the solution type of the infima, which is, conveniently, precisely the option describing wormhole solutions.

We would like to mention at this point that even though the introduction of the phantom field term $\pi_0/(re^T)$ might seem like a rather small change in the equations (1.3a–1.3d), it turns out that this term destroys certain convenient properties that the pure EYM system has (particularly related to the monotonicity of the dependent variables), and various aspects of the proofs become considerably more difficult. In particular, the radius function r could display oscillatory behaviour, which is generally difficult to treat. This somewhat resembles the situation Maison faced in [Mai05] when studying the apparently simpler $\mathbf{SU}(2)$ Yang-Mills-dilaton system using the methods developed for the pure EYM system.¹

Moreover, the reader should note that only the even solutions (i.e. with even n) from our main theorem are in analogy with black hole solutions in pure $\mathbf{SU}(2)$ EYM theory. The reason for this discrepancy comes from the fact that the odd solutions (i.e. with odd n) from our main theorem require initial conditions which are not compatible with the pure EYM equations [BFM94; Hau+14], and consequently such solutions were not studied in that setting. The existence proof for the odd case actually turns out to be surprisingly more difficult than for the even case, and in fact requires new techniques developed specifically for the wormhole initial conditions.

¹To quote Maison [Mai05]: “In [...] a classification of all static, spherically symmetric solutions of the EYM theory with a regular origin was given; our aim is to prove a corresponding classification for the YMD theory. Naively one could expect that this should be an easier task for the YMD theory due to its simpler structure. However, life is not so simple and, although ultimately the result is the same, the proof for the YMD theory appears to be more difficult.”

To conclude the summary, we would like to mention that our numerical analysis of the equation system (1.3a–1.3d) also indicates the existence of asymmetric wormhole solutions of the haunted $\mathbf{SU}(2)$ EYM equations. At the end of the manuscript we present an idea on how a proof of their existence may be conducted, although we have not been able to complete the proof mathematically. The existence of asymmetric haunted $\mathbf{SU}(2)$ EYM wormholes is therefore left merely as a conjecture, for now.

1.4 Organization of the manuscript

For the reader’s convenience, let us provide a short overview of the manuscript, as well as a short guide for reading it.

In §2 we discuss spacetime holes in spherically symmetric manifolds. In particular, in §2.2 we provide a local comparison of wormholes and black/white holes, and also discuss some energy conditions. We then specialize to the static setting in §2.3 and finally reach a definition of wormholes that will be used throughout the work. This chapter (in particular §2.2) is mostly motivational, so the reader not interested in too many details in this context can just look at Definition 2.3.2 and skip to future chapters without losing too much continuity.

In §3 we provide a brief review of principal bundles as a preparation for the Yang-Mills aspects of the haunted Einstein-Yang-Mills theory. The reader familiar with principal bundle theory may wish to skip the preliminaries in §3.1 entirely. In §3.2 we study spherically symmetric connections on principal $\mathbf{SU}(2)$ and derive the most general form of such a connection, which will serve as an ansatz for our Yang-Mills theory. The ansatz is fairly well-known in research articles, but the details for deriving it seem to be somewhat difficult to find, so we provide a detailed derivation. If one wishes to skip these details, one could just read the section starting from Theorem 3.2.5.

In §4, we formulate the haunted Einstein-Yang-Mills theory as a local Lagrangian theory, and derive the corresponding Euler-Lagrange equations in Theorem 4.1.2. We then specialize these equations to the static spherically symmetric setting in §4.2. The reader not interested in the general formulation of the theory can just look at the final equation system, contained in Theorem 4.2.1, as this is the system that will be studied throughout the rest of the manuscript. We then proceed to discuss appropriate wormhole conditions in §4.3 and derive some classical wormhole solutions in §4.4. Finally, we prove a “no-hair” result in §4.5, which considerably simplifies the system.

§5 is the heart of the thesis and contains the main results of the thesis discussed above in §1.3. This chapter also mirrors the contents of the author’s preprint [Sob23]. It is for the most part self-contained, so the reader interested only in the main results of the thesis could just read this chapter directly and refer to the previous chapters as necessary.

Finally, in §6 we discuss some additional problems which are related to the project but not properly addressed in this manuscript. We also present some possible generalizations of the theory, which could be interesting in future studies.

Chapter 2

Geometry of spacetime holes

In this chapter, we discuss the theory of spacetime holes. We first briefly recall some notions from semi-Riemannian geometry in §2.1, and we define spherically symmetric spacetimes. In §2.2 we define wormhole mouths and black/white hole horizons in terms of trapping horizons, which also allows for a simple comparison between these types of holes. Finally, in §2.3, we specialize to the static setting and in particular fix the definition of wormholes that will be used throughout the rest of the work.

2.1 Preliminaries

We will assume that the reader is familiar with differential geometry. Throughout the chapter, we let M be a smooth manifold of dimension n , unless otherwise specified. We denote by $\mathfrak{X}(M) = \Gamma(TM)$ the space of smooth vector fields on M , and by $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$ the space of differential k -forms on M .

2.1.1 Semi-Riemannian geometry

Let g be a semi-Riemannian metric on M , i.e. a symmetric non-degenerate tensor field $g \in \Gamma(T^*M \otimes T^*M)$ of constant index (i.e. the number of negative eigenvalues of g). In the particular case when the index is equal to one, we say that g is a Lorentzian metric. By convention, a vector field $X \in \mathfrak{X}(M)$ is said to be:

- ◇ *timelike* if $g(X, X) < 0$,
- ◇ *null* if $g(X, X) = 0$,
- ◇ *spacelike* if $g(X, X) > 0$.

The category which X falls into is called the *causal character* of X . If g is Lorentzian, we say that (M, g) is *time-orientable* if it admits a global timelike vector field $T \in \mathfrak{X}(M)$, and a choice of such a vector field is called a *time-orientation*. A vector field $X \in \mathfrak{X}(M)$ which is timelike is in this case said to be *future-directed* (resp. *past-directed*) if

$$g(X, T) < 0 \quad (\text{resp. } g(X, T) > 0).$$

By continuity we can also extend this notion to the case when X is null.

One can also define the notion of causal character for submanifolds of M . In the particular case of a hypersurface $\Sigma \subset M$ (i.e. a submanifold of codimension one) of a Lorentzian manifold (M, g) , one can define the causal character simply in terms of its normal vector ν , so that Σ is said to be

- ◊ *timelike* if ν is spacelike along Σ ,
- ◊ *null* if ν is null along Σ ,
- ◊ *spacelike* if ν is timelike along Σ .

Symmetries of a semi-Riemannian manifold (M, g) are often stated in terms of the existence of *Killing fields*, which are defined as vector fields $X \in \mathfrak{X}(M)$ satisfying the Killing equation

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}$ denotes the Lie derivative.

We define by $\sharp : T^*M \rightarrow TM$ the usual musical isomorphism characterized by

$$g(\eta^\sharp, X) = \eta(X), \quad \eta \in \Gamma(T^*M), \quad X \in \Gamma(TM),$$

and by $\flat : TM \rightarrow T^*M$ its inverse. For example, the gradient of a differentiable function $\phi \in C^1(M)$ is defined as the vector field $\nabla\phi = (d\phi)^\sharp$. The musical isomorphisms allow us to define inner product for one forms $\eta, \theta \in \Omega^1(M)$ by

$$\langle \eta, \theta \rangle_g = g(\eta^\sharp, \theta^\sharp),$$

in terms of which the causal character of a one-form can equivalently be determined. In local coordinates x^μ , this reads

$$(dx^\mu)^\sharp = g^{\mu\nu} \frac{\partial}{\partial x^\nu}, \quad \langle dx^\mu, dx^\nu \rangle_g = g^{\mu\nu}.$$

This induces a metric on the space $\Omega^k(M)$ of differential k -forms on M , explicitly given by

$$\langle \theta^1 \wedge \dots \wedge \theta^k, \eta^1 \wedge \dots \wedge \eta^k \rangle_g = \det \langle \theta^\mu, \eta^\nu \rangle_g, \quad (2.1)$$

where $\theta^\mu, \eta^\nu \in \Omega^1(M)$.

If (M, g) is oriented, we also define the *volume form* in a positively oriented coordinate system via

$$\text{vol}_g = \sqrt{|\det g|} dx^1 \wedge \dots \wedge dx^n.$$

The *Hodge star operator* $\star : \Omega^k(M) \rightarrow \Omega^{n-k}(M)$ on k -forms on M is the linear operator characterized by the relation

$$\eta \wedge \star \theta = \langle \eta, \theta \rangle \text{vol}_g,$$

where $\theta, \eta \in \Omega^k(M)$.

The *Einstein field equation* is the tensorial equation

$$\text{Ric}_g - \frac{1}{2}R_g g = T, \quad (2.2)$$

where Ric_g is the Ricci tensor, $R_g = \text{tr}_g(\text{Ric}_g)$ is the scalar curvature, and T is the so-called *stress-energy tensor*, which is usually derived from a Lagrangian theory (we will study this in more detail later). The Einstein field equation has an equivalent *trace-reversed* formulation

$$\text{Ric}_g = T - \frac{1}{2} \text{tr}_g(T) g,$$

as can be seen by taking the trace in (2.2) and plugging it back into the equation.

2.1.2 Static and spherically symmetric spacetimes

Throughout this manuscript, we will focus on spherically symmetric spacetimes, which we define as follows.

Definition 2.1.1. A *spherically symmetric spacetime* is a time-oriented four-dimensional Lorentzian manifold (M, g) such that

$$M \cong N \times \mathbb{S}^2, \quad g = h + r^2 g_{\mathbb{S}^2},$$

where

- ◊ (N, h) is a two-dimensional connected Lorentzian manifold,
- ◊ r is a smooth positive function on N , called the *spherical radius*,
- ◊ $g_{\mathbb{S}^2}$ is the round metric on $\mathbb{S}^2$, given in standard spherical coordinates (θ, φ) by

$$g_{\mathbb{S}^2} = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi.$$

Thus, a spherically symmetric spacetime can be viewed as the warped product

$$(M, g) = (N, h) \times_r (\mathbb{S}^2, g_{\mathbb{S}^2}).$$

Note that the Lie group $\mathbf{SO}(3)$ acts naturally on (M, g) (in particular on the spherical part) by isometries. Equivalently, the vector fields

$$\frac{\partial}{\partial \varphi}, \quad -\sin \varphi \frac{\partial}{\partial \theta} - \cot \theta \cos \varphi \frac{\partial}{\partial \varphi}, \quad \cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi}$$

are Killing fields on (M, g) .

Remark. If one would like to study less trivial topologies, one could more generally define a spherically symmetric spacetime (M, g) as a fiber bundle (cf. §3.1.1) $\mathbb{S}^2 \rightarrow M \xrightarrow{\pi} N$ such that the action of the isometry group $\mathbf{SO}(3)$ on $(\mathbb{S}^2, g_{\mathbb{S}^2})$ induces an action on (M, g) by isometries (see [Kün67] for more details). However, one then has the decomposition $M \cong N \times \mathbb{S}^2$ only locally, so we prefer to keep our definition for simplicity.

Observe that r is a geometric invariant. Indeed, for a given point $x \in N$, we may consider the sphere $\mathbb{S}_x^2 := \{x\} \times \mathbb{S}^2 \subset M$ at $x \in N$, and then r is given in terms of the volume of $\mathbb{S}_x^2$ (corresponding to the metric induced by g) by

$$r(x) = \sqrt{\frac{\text{vol}_g(\mathbb{S}_x^2)}{\text{vol}(\mathbb{S}^2)}} = \sqrt{\frac{\text{vol}_g(\mathbb{S}_x^2)}{4\pi}}.$$

There is another important geometric invariant in spherical symmetry, called the *mass* (or *gravitational energy*) $m : N \rightarrow \mathbb{R}$, defined by

$$m = \frac{r}{2}(1 - \langle dr, dr \rangle). \quad (2.3)$$

This quantity originates from Misner and Sharp [MS64] and is thus often also called the Misner-Sharp mass. It also agrees with the Hawking mass, which is usually defined for general closed two-dimensional spacelike surfaces, not necessarily just spheres. Provided that the spacetime is asymptotically flat, the Misner-Sharp mass coincides with the Arnowitt-Deser-Misner (ADM) mass at spacelike infinity and the Bondi mass at null infinity [Hay96, Proposition 3]. In the future sections, we will in fact take the finiteness of the mass at infinity as a defining property of asymptotically flat manifolds.

To end this section, we also recall the definition of static Lorentzian manifolds.

Definition 2.1.2. A Lorentzian manifold (M, g) is said to be *static* if it admits a timelike Killing field $T \in \mathfrak{X}(M)$ and foliation by spacelike hypersurfaces orthogonal to T .

For a static spherically symmetric spacetime $(M, g) = (N, h) \times_r (\mathbb{S}^2, g_{\mathbb{S}^2})$, we may locally choose coordinates (t, ρ) on N such that the timelike Killing field is given by $T = \partial_t$, the radius function r only depends on ρ , and the metric h takes the form

$$h = -e^{2\tau(\rho)} dt \otimes dt + e^{2\sigma(\rho)} d\rho \otimes d\rho,$$

for some functions τ, σ depending only on ρ (see also [ONe83, §13]). There is an additional freedom in rescaling the ρ -coordinate, which we will use (for later convenience) to rescale $e^{\sigma(\rho)} d\rho \mapsto r(\rho) d\rho$, so that the metric becomes

$$g = -e^{2\tau(\rho)} dt \otimes dt + r(\rho)^2 (d\rho \otimes d\rho + g_{\mathbb{S}^2}). \quad (2.4)$$

Here, ρ can be viewed as a generalization of the so-called *log-polar radius*.¹ This choice of coordinates is the one that we will be working with throughout most of the manuscript (except in the next section). Note that the mass (2.3) is then given simply by

$$m = \frac{r}{2} \left(1 - \frac{\dot{r}^2}{r^2} \right), \quad (2.5)$$

where overdot denotes differentiation with respect to ρ .

¹A common alternative choice of radial coordinate for describing static spherically symmetric worm-holes is the so-called *proper radius* defined via $d\ell = r d\rho$, in which case the metric is given by

$$g = -e^{2\tau(\ell)} dt \otimes dt + d\ell \otimes d\ell + r(\ell)^2 g_{\mathbb{S}^2}.$$

2.2 Spacetime holes

In this section, we discuss spacetime holes à la Hayward [Hay09; Hay99]. These are defined in terms of so-called trapping horizons, which allows one to locally unify the theory of black/white holes and wormholes. Traditionally, black holes are teleological objects since their commonly accepted definition requires the knowledge of the causal structure of spacetime arbitrarily far into the future. The definition we employ here has the benefit of being local while still reasonably capturing properties that a black/white hole is expected to satisfy. Even more importantly in the context of this work, the definition allows for a relatively simple comparison with wormholes. It should however be noted that, in contrast to black holes, there is no universally accepted definition of a (dynamic) wormhole, and the definition presented here is merely a reasonable suggestion.

2.2.1 Trapping horizons

Let $M = N \times \mathbb{S}^2$ be a spherically symmetric spacetime (not necessarily static) with spherical radius r and mass m . Define the sphere at $x \in N$ as $\mathbb{S}_x^2 = \{x\} \times \mathbb{S}^2$. We say that the sphere $\mathbb{S}_x^2$ is

- ◇ *untrapped* if $r > 2m$ at x , i.e. if $\nabla r(x)$ is spacelike,
- ◇ *marginal* if $r = 2m$ at x , i.e. if $\nabla r(x)$ is null,
- ◇ *trapped* if $r < 2m$ at x , i.e. if $\nabla r(x)$ is timelike.

A hypersurface $\Sigma \subset M$ is said to be a *trapping horizon* if it is foliated by marginal spheres, i.e. if

$$\Sigma = \Gamma \times \mathbb{S}^2 \quad \text{with} \quad r = 2m \quad \text{along} \quad \Gamma.$$

We say that a trapping horizon Σ is *future* (resp. *past*) if ∇r is future-directed (resp. past-directed) along Γ , and otherwise it is said to be *static*.

An important quantity related to trapping horizons is the so-called *surface gravity*, defined as

$$\kappa = \frac{1}{2} \star d \star dr = \frac{1}{2} \square_h r,$$

where $\star$ denotes the Hodge star operator on (N, h) and $\square_h = h^{\mu\nu} \nabla_\mu \partial_\nu$ is the wave operator. Note that this agrees with the usual definition of surface gravity for Killing horizons [Hay09, (13)]. In terms of the surface gravity, we classify trapping horizons $\Sigma = \Gamma \times \mathbb{S}^2$ as

- ◇ *outer* if $\kappa > 0$ along Γ ,
- ◇ *degenerate* if $\kappa = 0$ along Γ ,
- ◇ *inner* if $\kappa < 0$ along Γ .

Definition 2.2.1 ([Hay09; Hay99]). An outer trapping horizon $\Sigma \subset M$ is said to be a

- ◇ *wormhole mouth* if Σ is timelike,

- ◇ *(apparent) black hole horizon* if Σ is future and either null or spacelike,
- ◇ *(apparent) white hole horizon* if Σ is past and either null or spacelike.

Remark. The outerness condition $\kappa > 0$ is made for here for simplicity. The degenerate case $\kappa = 0$ should also be admissible, but it is somewhat troublesome to do so in the general dynamic case (cf. [Hay09, §III]). We will, however, also allow for vanishing surface gravity once we start working with static spacetimes in the next section.

In terms of this definition, wormholes and black/white holes thus have locally indistinguishable geometries, except for the causal character of the trapping horizon that defines them. We therefore collectively call them *spacetime holes*. The reader should note that a wormhole mouth here describes only one side of the wormhole, and a complete wormhole spacetime should actually have two mouths with “opposite senses” (we will make this more precise below) and in mutual causal contact, so as to allow traversability.

We consider now two classical examples, which show that these definitions are reasonable.

Example 2.2.2. Consider the Ellis-Bronnikov metric

$$g = -dt \otimes dt + r_0^2 \cosh^2(\rho) (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

where $r_0 > 0$ is the so-called *throat radius*. In this case,

$$r(\rho) = r_0 \cosh(\rho) \quad \text{and} \quad m(\rho) = \frac{r_0}{2} \operatorname{sech}(\rho).$$

Thus the hypersurface

$$\mathcal{T} = \{\rho = 0\} \times \mathbb{S}^2$$

is a trapping horizon. Additionally, the surface gravity is given by

$$\kappa(\rho) = \frac{r_0}{2} \operatorname{sech}^3(\rho),$$

so $\kappa \equiv \frac{r_0}{2}$ on $\mathcal{T}$ and the trapping horizon $\mathcal{T}$ is outer. Finally, the spacelike vector field ∂_ρ is normal to $\mathcal{T}$, so $\mathcal{T}$ is timelike. Hence, $\mathcal{T}$ describes a wormhole mouth.

Example 2.2.3. The well-known Schwarzschild metric is given by

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2},$$

where $m > 0$ is the mass parameter (note that in this case, the mass (2.3) is constantly equal to the parameter m , so the notation is unambiguous). We expect that $r = 2m$ describes a black hole horizon. However, the classical Schwarzschildian coordinate system is not suitable to make such claims, because the metric has a coordinate singularity at $r = 2m$. We consider instead its maximal extension given by the Kruskal-Szekeres manifold in a double-null coordinate system by the metric

$$g = -\frac{16m^3}{r} \exp\left(-\frac{r}{2m}\right) (dx_+ \otimes dx_- + dx_- \otimes dx_+) + r^2 g_{\mathbb{S}^2},$$

where the coordinates $x_{\pm}$ are related to r via

$$x_+x_- = \left(1 - \frac{r}{2m}\right) \exp\left(\frac{r}{2m}\right).$$

A differentiation of this identity reveals that

$$dr = -\frac{4m^2}{r} \exp\left(-\frac{r}{2m}\right) (x_- dx_+ + x_+ dx_-),$$

from which we also deduce

$$\nabla r = (dr)^{\sharp} = \frac{1}{4m} (x_+ \partial_+ + x_- \partial_-), \quad (2.6)$$

where $\partial_{\pm} = \partial/\partial x_{\pm}$. Now

$$g(\nabla r, \nabla r) = \langle dr, dr \rangle_g = -\frac{2m}{r} \exp\left(-\frac{r}{2m}\right) x_+x_-,$$

and it follows that the null hypersurfaces

$$\begin{aligned} \mathcal{H}^+ &= (\{x_+ = 0, x_- > 0\} \cup \{x_+ > 0, x_- = 0\}) \times \mathbb{S}^2, \\ \mathcal{H}^- &= (\{x_+ = 0, x_- < 0\} \cup \{x_+ < 0, x_- = 0\}) \times \mathbb{S}^2 \end{aligned}$$

are trapping horizons. Observe that the surface gravity satisfies

$$\begin{aligned} \kappa &= \frac{1}{2} \star d \star dr \\ &= -\star d \left(\frac{2m^2}{r} \exp\left(-\frac{r}{2m}\right) (x_- dx_+ - x_+ dx_-) \right) \\ &= \star(dx_+ \wedge dx_-) \frac{4m^2}{r} \exp\left(-\frac{r}{2m}\right) \left[1 + x_+x_- \left(1 + \frac{2m}{r} \right) \exp\left(-\frac{r}{2m}\right) \right] \\ &= \frac{1}{4m} \left[1 + x_+x_- \left(1 + \frac{2m}{r} \right) \exp\left(-\frac{r}{2m}\right) \right], \end{aligned}$$

from which it follows that $\kappa \equiv \frac{1}{4m}$ on $\mathcal{H}^{\pm}$ and therefore these are outer trapping horizons. Finally, if we time-orient the manifold as usual by declaring the vector fields $\partial_{\pm}$ to be future-oriented, then we see from (2.6) that $\mathcal{H}^+$ is future and $\mathcal{H}^-$ is past. Therefore $\mathcal{H}^+$ describes a black hole horizon and $\mathcal{H}^-$ describes a white hole horizon.

In view of the last example, it may not come as a surprise that it is convenient to study trapping horizons by expressing the metric h in a double-null coordinates (x_+, x_-) as

$$h = -e^{2\gamma(x_+, x_-)} (dx_+ \otimes dx_- + dx_- \otimes dx_+).$$

This is possible (at least locally) since every two-dimensional Lorentzian manifold is conformally flat [Sch08, Theorem 7.2]. We assume also that (N, h) is time-oriented so that both $\partial_{\pm}$ are future-oriented.

Observe that

$$g(\nabla r, \nabla r) = \langle dr, dr \rangle_g = -2e^{-2\gamma} \partial_+ r \partial_- r,$$

where hereafter we denote $\partial_{\pm} = \partial/\partial x_{\pm}$. It follows that on a trapping horizon, at least one of the derivatives $\partial_{\pm} r$ must vanish. If, without loss of generality, we have $\partial_+ r = 0$ on a trapping horizon Σ , then we see from

$$\nabla r = (dr)^{\sharp} = \partial_- r (dx_-)^{\sharp} = -e^{-2\gamma} \partial_- r \partial_+ \quad \text{on } \Gamma,$$

that Σ is future (resp. past) if $\partial_- r < 0$ (resp. $\partial_- r > 0$) and static if $\partial_- r = 0$. On the other hand, the surface gravity is given in these coordinates by

$$\kappa = -e^{-2\gamma} \partial_- \partial_+ r \quad \text{on } \Gamma,$$

so the outer condition $\kappa > 0$ demands $\partial_- \partial_+ r < 0$.

Now, if we take a non-zero tangent vector field $X \in \mathfrak{X}(\Gamma)$ and write

$$X = X_+ \partial_+ + X_- \partial_-,$$

then since $\partial_+ r = 0$ along Γ , we get

$$0 = (d\partial_+ r)(X) = X_+ \partial_+ \partial_+ r + X_- \partial_- \partial_+ r. \quad (2.7)$$

This implies the following.

- ◊ If Σ is a wormhole mouth, then X must be timelike, which implies $X_+ X_- > 0$. Hence,

$$\partial_+ \partial_+ r = -\frac{X_-}{X_+} \partial_- \partial_+ r > 0,$$

by the outerness condition. This expresses the fact that the spheres at a mouth are minimal in the x^+ -direction, and such minimality behaviour is precisely what we wish to capture when describing a wormhole mouth.

- ◊ If Σ is a black/white hole horizon, then X is null or spacelike, i.e. $X_+ X_- \leq 0$. Then

$$\partial_+ \partial_+ r = -\frac{X_-}{X_+} \partial_- \partial_+ r \leq 0,$$

again by the outerness condition, where we additionally note that X_+ cannot be zero, since then (2.7) would be a contradiction for any non-zero X_- (in fact, we then see that in the null case we must have $X_- = 0$).

We would like to end this section by pointing out that, as already mentioned, a complete wormhole region of a spacetime should actually have two such mouths, marginal in opposite null directions (Hayward captures this by saying the mouths should have “opposite senses”), and also in mutual causal contact, in the sense that any pair of points from the two mouths can be connected by a causal curve. This latter conditions ensures that the resulting wormhole region is two-way traversable, so an observer can enter one wormhole mouth and exit through the other. In fact, we will see that in the static case, the two mouths will be described by a single trapping horizon which is marginal in both null directions simultaneously (being timelike, such a trapping horizon is of course trivially in causal contact with itself).

2.2.2 Energy conditions

Let us continue by discussing some energy conditions. There are many different energy conditions commonly employed in general relativity, see e.g. [Vis96, §12.1] for a comprehensive review. These are usually defined in terms of the stress-energy tensor T , which can in fact be viewed as a geometric invariant of a spacetime owing to the Einstein field equation (2.2). Possibly the most famous energy conditions are the null (NEC), weak (WEC), strong (SEC), and dominant (DEC) energy conditions, defined respectively via

$$\begin{aligned} \text{(NEC)} \quad &\iff T(X, X) \geq 0 \text{ for any null vector field } X, \\ \text{(WEC)} \quad &\iff T(X, X) \geq 0 \text{ for any timelike vector field } X, \\ \text{(SEC)} \quad &\iff T(X, X) - \frac{1}{2} \text{tr}(T) g(X, X) \geq 0 \text{ for any timelike vector field } X, \\ \text{(DEC)} \quad &\iff T(X, X) \geq 0 \text{ and } T(X, \cdot) \text{ is not spacelike for any timelike } X. \end{aligned}$$

For example, the positive mass theorem of Schoen and Yau assumes the DEC [SY79], while the NEC is hypothesised in the classical Penrose singularity theorem [HE73, §8.2]. Note that the NEC is the weakest of the four conditions, as can be seen by approximating a null vector by a sequence of timelike vectors. Matter violating the NEC is usually referred to as *exotic matter*.

For the double-null frame $\partial_{\pm}$ introduced above, we get that

$$T(\partial_{\pm}, \partial_{\pm}) = \text{Ric}(\partial_{\pm}, \partial_{\pm}) = \frac{2\partial_{\pm}\gamma\partial_{\pm}r - \partial_{\pm}\partial_{\pm}r}{2r},$$

or, on a trapping horizon (again without loss of generality with $\partial_+r = 0$)

$$T(\partial_+, \partial_+) = -\frac{\partial_+\partial_+r}{2r}.$$

In particular, since a wormhole mouth with $\partial_+r = 0$ has $\partial_+\partial_+r > 0$ along Γ , we see that the NEC must be violated on a wormhole mouth. Therefore, a wormhole requires support from exotic matter for its existence. Note that, in contrast, the null/spacelike character of a black hole horizon is not in contradiction with the NEC.

2.3 Static spherically symmetric wormholes

In contrast to the dynamic setting from the previous section, there in fact exists a generally accepted definition of a wormhole in the static setting. Such wormholes were introduced mathematically in the pioneering work of Morris and Throne [MT88], and subsequently studied by many other authors, see e.g. the extensive book of Visser [Vis96] on the subject. In this section we will study static spherically symmetric wormholes at a global level and compare their definition to the one from the previous section.

For a static spherically symmetric spacetime $(M, g) = (N, h) \times_r (\mathbb{S}^2, g_{\mathbb{S}^2})$, we can select coordinates (t, ρ) on N so that the metric is given by

$$g = -e^{2\tau(\rho)} dt \otimes dt + r(\rho)^2 (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

for some function τ , cf. §2.1.2. One can easily relate this coordinate system to a double-null coordinate system as in §2.2, e.g. by defining²

$$x_{\pm} = t \pm \int_0^{\rho} r e^{-\tau}.$$

We will henceforth assume that $N \cong \mathbb{R}^2$ and that (t, ρ) are global coordinates on N . As before, we will denote differentiation with respect to ρ by an overdot, and we set $\partial_{\pm} = \partial/\partial x_{\pm}$.

We recall from (2.5) that the mass m is given by

$$m = \frac{r}{2} \left(1 - \frac{\dot{r}^2}{r^2} \right).$$

We therefore directly see that the characterizing condition $r = 2m$ of trapping horizons is equivalent to $\dot{r} = 0$. Thus the hypersurface $\Sigma = \{\rho = \rho_0\}$ is a trapping horizon provided that ρ_0 is a critical point of r . In terms of the double-null coordinates, we have $\partial_+ r = \partial_- r = 0$, so such a trapping horizon is marginal in both null-directions, i.e. it is a static trapping horizon. Note that such a trapping horizon is automatically timelike since the spacelike vector field ∂_{ρ} is normal to Σ . The surface gravity is given by

$$\kappa = \frac{1}{2r^2} \left[\left(\dot{r} - \frac{\dot{r}}{r} \right) \dot{r} + \ddot{r} \right].$$

Evaluating at Σ , we therefore see that the sign of κ is determined by $\ddot{r}$. In particular, Σ describes a (non-degenerate) wormhole mouth if and only if $\ddot{r} > 0$. In terms of the double-null coordinates, this we have $\partial_+ \partial_+ r = \partial_- \partial_- r > 0$, and thus the spheres at the mouth are minimal in both of the null directions. Thus the radius function r has a minimum at a wormhole mouth, as expected.

We would now like to define a wormhole spacetime as a global model, in particular as a tunnel connecting two asymptotically flat ends. For static spacetimes, one defines asymptotic flatness by requiring that the spacelike slices $\{t = \text{const.}\}$ are asymptotically flat in the Riemannian sense. In the spherically symmetric framework, the latter is most conveniently defined in terms of the mass (2.5), see e.g. [BP92].

Definition 2.3.1. A static spherically symmetric spacetime (cf. Definition 2.1.1) is said to be *asymptotically flat with two ends* if

$$\lim_{\rho \rightarrow \pm\infty} r(\rho) = \infty, \quad \lim_{\rho \rightarrow \pm\infty} m(\rho) = m_{\infty}^{\pm}, \quad \lim_{\rho \rightarrow \pm\infty} \tau(\rho) = \tau_{\infty}^{\pm},$$

where $m_{\infty}^{\pm}, \tau_{\infty}^{\pm} \in \mathbb{R}$ are finite.

²We would like to note at this point that the static coordinates (t, ρ) are not really suitable for studying black/white hole horizons, as the metric will generally become degenerate in these coordinates. This can be seen already in the Schwarzschild case, where the classical coordinates are ill-behaved at the horizon, and one is inclined to use e.g. the Kruskal-Szekeres coordinates as in Example 2.2.3, cf. also Table 4.4. Nevertheless, we will henceforth only focus solely on wormholes, which will be globally regular in both the (t, ρ) and the (x_+, x_-) system.

Remark. These conditions ensure that the induced metric on the spacelike slices $\{t = \text{const.}\}$ satisfies

$$g_{ij} = \delta_{ij} + O(1/r)$$

in the standard Cartesian coordinates x^i . One could obtain decay rates for the derivatives of the metric g_{ij} by imposing additional boundedness assumptions on the derivatives of the mass m , but we do not include this in the general definition.

Note that the property $r \rightarrow \infty$ as $\rho \rightarrow \pm\infty$ actually implies that there exists *at least one* point $\rho = \rho_0$ at which r has a (local) minimum, which must also be positive since $r > 0$ throughout the spacetime. At such a point, we necessarily have

$$\dot{r}(\rho_0) = 0, \quad \ddot{r}(\rho_0) \geq 0,$$

so that $\rho = \rho_0$ describes a timelike trapping horizon which is outer if $\ddot{r}(\rho_0) > 0$, or degenerate if $\ddot{r}(\rho_0) = 0$. In the degenerate case, such a trapping horizon should certainly also be considered a wormhole mouth in the static case [Vis96, §11.2]. In particular, we see that the global requirement of asymptotic flatness at $\pm\infty$ automatically implies the local definition of a wormhole mouth, provided that we allow it to have vanishing surface gravity.

More generally, r could have multiple critical points, not all of which need to be minima. To avoid overly pathological examples, we shall make the additional assumption that r has only finitely many critical points, so that r is piecewise monotone. To complete the nomenclature, we then say that a critical point ρ_0 of r describes:³

- ◊ a *wormhole throat* if r has a local minimum at ρ_0 ,
- ◊ a *wormhole belly* if r has a local maximum at ρ_0 .

Thus when the surface gravity at ρ_0 is positive, we get a wormhole throat, and when the surface gravity at ρ_0 is negative, we get a wormhole belly. In the degenerate case when the surface gravity vanishes, we have $\ddot{r}(\rho_0) = 0$ and ρ_0 could describe either a throat or a belly, or it could even be a non-extremal point of r (this was one of the main reasons for excluding the degenerate case in the previous section, as the picture is considerably more complicated when the spacetime is not static).

The discussion above allows for the following simple definition, which is essentially equivalent to the standard definition [Vis96, §11.2], the only difference being that we also allow for multiple throats.

Definition 2.3.2. A *wormhole* is a static spherically symmetric spacetime $\mathbb{R}^2 \times \mathbb{S}^2$ equipped with a Lorentzian metric of the form

$$g = -e^{2\tau(\rho)} dt \otimes dt + r(\rho)^2 (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

satisfying the following conditions:

³Such nomenclature is common when dealing with static wormholes. In the non-static cases, it would seem that a wormhole throat has several different meanings for different authors, so we preferred to use the term mouth as suggested by Hayward [Hay09].

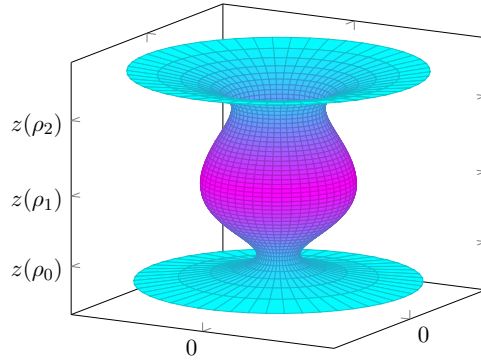


Figure 2.1: Embedding diagram of a generic wormhole with throats at $\rho = \rho_0$, $\rho = \rho_2$ and a belly at $\rho = \rho_1$.

- ◇ $\tau, r \in C^\infty(\mathbb{R})$ with $r > 0$ everywhere and r has only finitely many critical points;
- ◇ g is asymptotically flat with two ends, in the sense of Definition 2.3.1.

If, in addition, we have that

- ◇ τ and r are even functions of ρ ,

then the wormhole is said to be *symmetric*.

Remark. This definition is purely geometric. From a physics perspective however, one usually also wishes to impose that g obeys the Einstein field equations (2.2) for some accepted matter model, as we will also do in the future chapters.

It is common visualize a wormhole by freezing the time coordinate and fixing $\theta = \pi/2$, so that one looks at the equatorial slice, which is described by the surface of revolution with induced metric

$$r(\rho)^2 (d\rho \otimes d\rho + d\varphi \otimes d\varphi).$$

Such a surface can be parametrized in $\mathbb{R}^3$ via

$$(\rho, \varphi) \mapsto (r(\rho) \cos \varphi, r(\rho) \sin \varphi, z(\rho)),$$

where z is given by

$$z(\rho) = \int_0^\rho \sqrt{r^2 - \dot{r}^2},$$

as long as the right-hand side is well-defined (note that it is well defined at least near a throat since $r > 0$ and $\dot{r} = 0$ at any throat). This is usually referred in literature as the *embedding diagram* of the wormhole, an example of which we have already seen in Figure 1.1 for the Ellis-Bronnikov wormhole. We present also a more generic example of an embedding diagram in Figure 2.1.

Chapter 3

Geometry of principal bundles

In this chapter, we provide a brief overview of the tools necessary for the understanding of Yang-Mills theory. We then study principal $\mathbf{SU}(2)$ -bundles with symmetry in more detail, and in particular derive a well-known ansatz for the gauge potential, which will be used in future chapters.

3.1 Principal bundle theory

Let us first discuss our notation concerning differential forms taking values in a fiber bundle in §3.1.1. We will then provide a brief review of principal bundle theory in §3.1.2. For more details on the topic, as well as elementary Lie theory, the reader can refer e.g. to [KN63; Ble81].

3.1.1 Bundle-valued forms

Let M, E, F be smooth manifolds. A *fiber bundle* over M with total space E and fiber F (often denoted $F \rightarrow E \xrightarrow{\pi} M$) is a smooth surjection $\pi : E \rightarrow M$ such that for each $x \in M$, there is an open neighbourhood U containing x and a diffeomorphism

$$\phi : \pi^{-1}(U) \rightarrow U \times F$$

which preserves each fiber, in the sense that the restriction of ϕ to the set $\pi^{-1}(x)$ is also a diffeomorphism onto $\{x\} \times F$ for each $x \in U$. We call ϕ a *local trivialization*. A (local) *section* of the fiber bundle E is a smooth map $\sigma : U \subset M \rightarrow E$ such that $\pi \circ \sigma$ is the identity map on $U \subset M$. The space of (local) sections of E is denoted by $\Gamma(E) = \Gamma_U(E)$, where we usually suppress the open domain U if it is clear from the context.

For a fiber bundle E over M , we denote by

$$\Omega^k(M, E) = \Gamma(\Lambda^k T^*M \otimes E)$$

the space of E -valued differential k -forms on M . If $E = \mathbb{R}$, we obtain the usual real-valued differential forms on M , and simply use the notation $\Omega^k(M)$. We can define various familiar operations on E -valued k -forms.

- ◇ The *exterior derivative* $d : \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)$ is defined by

$$d(\theta \otimes \sigma) = (d\theta) \otimes \sigma,$$

where $\theta \in \Omega^k(M)$, $\sigma \in \Gamma(E)$ and extended linearly.

- ◇ If $\phi : M \rightarrow N$ is a smooth map, the *pullback* by ϕ is the map $\phi^* : \Omega^k(N, E) \rightarrow \phi^*\Omega^k(N, E)$ defined by

$$\phi^*(\theta \otimes \sigma) = (\phi^*\theta) \otimes \sigma$$

where $\theta \in \Omega^k(N)$, $\sigma \in \Gamma(E)$ and extended linearly. Here,

$$\phi^*\Omega^k(N, E) = \Gamma(\Lambda^k(\phi^*T^*N) \otimes E), \quad \phi^*T^*N = \coprod_{x \in M} T_{\phi(x)}^*N$$

and the pullback of the k -form θ is defined as usual by the formula

$$(\phi^*\theta)_x(X_1, \dots, X_k) = \theta_{\phi(x)}(d\phi_x(X_1), \dots, d\phi_x(X_k)),$$

for $x \in M$ and $X_j \in T_xM$.

- ◇ The *wedge product* $\wedge : \Omega^k(M) \times \Omega^\ell(M, E) \rightarrow \Omega^{k+\ell}(M, E)$ is defined by requiring

$$\theta \wedge (\varphi \otimes \sigma) = (\theta \wedge \varphi) \otimes \sigma$$

for all $\theta \in \Omega^k(M)$, $\varphi \in \Omega^\ell(M)$ and $\sigma \in \Gamma(E)$, and extending (bi)linearly. Thus we may write $\theta \wedge \varphi \otimes \sigma$ without ambiguity. Note that such an operation does not make sense if the first factor is also E -valued, unless we have additional structure.

- ◇ If $E = \mathfrak{g}$ has a Lie algebra structure, we also have the *commutator wedge product* $[\cdot \wedge \cdot] : \Omega^k(M, \mathfrak{g}) \times \Omega^\ell(M, \mathfrak{g}) \rightarrow \Omega^{k+\ell}(M, \mathfrak{g})$ given by

$$[(\theta \otimes \sigma) \wedge (\eta \otimes \lambda)] = (\theta \wedge \eta) \otimes [\sigma, \lambda]_{\mathfrak{g}},$$

for $\theta \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ and $\sigma, \lambda \in \mathfrak{g}$, and extended bilinearly. Note that this is a symmetric operation, because the antisymmetries of the wedge and commutator operations cancel out.

- ◇ If E comes equipped with a bundle metric, we may define a metric on $\Omega^k(M, E)$ via

$$\langle \theta \otimes \sigma, \eta \otimes \lambda \rangle = \langle \theta, \eta \rangle_g \langle \sigma, \lambda \rangle_E$$

for $\theta, \eta \in \Omega^k(M)$ and $\sigma, \lambda \in \Gamma(E)$, and extending bilinearly.

- ◇ The *metric wedge product* $\langle \cdot \wedge \cdot \rangle : \Omega^k(M, E) \times \Omega^\ell(M, E) \rightarrow \Omega^{k+\ell}(M)$ by

$$\langle (\theta \otimes \sigma) \wedge (\eta \otimes \lambda) \rangle = \langle \sigma, \lambda \rangle_E (\theta \wedge \eta),$$

for $\theta \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ and $\sigma, \lambda \in \Gamma(E)$, and extending bilinearly.

◇ The *Hodge star operator* $\star : \Omega^k(M, E) \rightarrow \Omega^{n-k}(M, E)$ is given by

$$\star(\theta \otimes \sigma) = (\star\theta) \otimes \sigma$$

for $\theta \in \Omega^k(M)$, $\sigma \in \Gamma(E)$ and extended linearly. Note that in terms of the metric wedge product, we then have

$$\langle \omega, \nu \rangle = \star \langle \omega \wedge \star \nu \rangle,$$

whenever $\omega, \nu \in \Omega^k(M, E)$.

3.1.2 Principal bundles

Let G be a Lie group and P, M smooth manifolds. A *principal G -bundle* $G \rightarrow P \xrightarrow{\pi} M$ is a fiber bundle $P \xrightarrow{\pi} M$ such that:

- ◇ G acts smoothly on P to the right,
- ◇ the action preserves the fibers and is regular along them,
- ◇ the local trivializations of the fiber bundle are G -equivariant.

The Lie group G is often called the *structure/gauge* group. We recall here that a (right) action of a group G on a manifold X is said to be *regular* if for any $x, y \in X$, there is a unique $g \in G$ such that $xg = y$. We also recall that a map $f : X \rightarrow Y$ between two manifolds on which G acts to the right is said to be *G -equivariant* if $f(xg) = f(x)g$ for all $x \in X$ and $g \in G$. These concepts are analogously defined also for left actions.

The local trivializations of a principal G -bundle $G \rightarrow P \xrightarrow{\pi} M$ are in correspondence with sections. Indeed, given a local section $\sigma : U \subset M \rightarrow P$, we can define

$$\psi : U \times G \rightarrow \pi^{-1}(U), \quad \psi(x, g) = \sigma(x)g,$$

whose inverses are local trivialization of P . Conversely, a trivialization $\phi : \pi^{-1}(U) \rightarrow U \times G$ defines a section $\sigma : U \rightarrow P$ via

$$\sigma(x) = \phi^{-1}(x, e).$$

In the context of principal G -bundles, it is often more convenient to work with sections than trivializations. A section is often also referred to as a *gauge*.

Let us give two examples.

- ◇ Possibly the simplest example of a principal G -bundle over a manifold M is the trivial bundle $G \rightarrow M \times G \xrightarrow{\pi} M$, where the projection and the action are given respectively by

$$\pi(x, g) = x, \quad (x, g)g_0 = (x, gg_0), \quad x \in M, \quad g, g_0 \in G.$$

This is referred to as the *trivial* principal G -bundle over M .

- ◊ A less trivial example of principal bundles is given by homogeneous spaces. Indeed, given a Lie group G and a closed Lie subgroup H , it is well-known that G/H defines a smooth manifold, and

$$H \rightarrow G \xrightarrow{\pi} G/H$$

defines a principal H -bundle, where the projection π is the usual quotient map $g \mapsto gH$, and the H acts trivially on G to the right.

The projection map π of a principal bundle $P \xrightarrow{\pi} M$ is a submersion, and each fiber $P_x = \pi^{-1}(\{x\})$ is a smooth manifold whose tangent space at $p \in P_x$ satisfies $T_p P_x = \ker d\pi_p$. The disjoint union of these tangent spaces forms a subbundle of TP called the *vertical distribution*

$$V = \ker d\pi = \coprod_{p \in P} \ker d\pi_p.$$

We note that V is right-invariant in the sense that $V_{pg} = dr_g(V_p)$. A k -form $\theta \in \Omega^k(P, \mathfrak{g})$ is said to be *horizontal* if it annihilates vertical vector fields, i.e. if

$$\theta(X_1, \dots, X_k) = 0 \tag{3.1}$$

whenever at least one of the vector fields $X_i \in \Gamma(TP)$ is vertical, i.e. an element of $\Gamma(V)$.

For $X \in \mathfrak{g}$, we define its *fundamental vector field* $X^* \in \Gamma(TP)$ as

$$X_p^* = \left. \frac{d}{dt} \right|_{t=0} p \exp(tX).$$

Note that the map

$$\mathfrak{g} \ni X \mapsto X_p^* \in V_p$$

is a linear isomorphism for each p .

A *connection form* on P is a $\mathfrak{g}$ -valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ which preserves vertical vector fields and is of Ad-type, i.e.

$$\begin{aligned} \omega(X^*) &= X, \quad \text{for } X \in \mathfrak{g}, \\ r_g^* \omega &= \text{Ad}_{g^{-1}} \circ \omega, \quad \text{for } g \in G. \end{aligned}$$

When G and $\mathfrak{g}$ are represented as matrix groups, the latter Ad-type condition is often rewritten using the notation

$$\text{Ad}_{g^{-1}} \circ \omega = g^{-1} \omega g,$$

where the right-hand side is interpreted as matrix multiplication on the $\mathfrak{g}$ -factor of ω . Given a section $\sigma : U \subset M \rightarrow P$, we can pull down the connection to a $\mathfrak{g}$ -valued 1-form $\sigma^* \omega = \omega^\sigma \in \Omega^1(U, \mathfrak{g})$ on U . The form thus obtained is often called a *gauge potential* and the section is referred to as a *gauge*. Given a map $g : V \rightarrow G$, the gauge transformation

induced by g is the transformation of the section via $\sigma \mapsto \sigma g$, under which the connection transforms as

$$\omega^\sigma \mapsto \text{Ad}_{g^{-1}} \circ \omega^\sigma + d\ell_{g^{-1}} \circ dg.$$

In fact, one could equivalently define a connection on the principal bundle by instead providing a family of sections and $\mathfrak{g}$ -valued forms respecting such a transformation law, and patching them to a well-defined form on P .

The transformation law features the important *Maurer-Cartan form* (sometimes also called the *canonical form*) $\mu = \mu^G \in \Omega^1(G, \mathfrak{g})$ of a Lie group G , defined as the identity map on $\mathfrak{g}$ and extended left-invariantly, i.e.

$$\mu_g = d\ell_{g^{-1}} \circ dg,$$

where we regard $g : G \rightarrow G$ as the identity mapping. If the Lie group is represented as a matrix group, then this is more commonly written as

$$\mu_g = g^{-1} dg,$$

where the right-hand side is interpreted as matrix multiplication on the $\mathfrak{g}$ factor of dg .

Given a connection $\omega \in \Omega^1(P, \mathfrak{g})$, the *exterior covariant derivative* with respect to ω is the map

$$d_\omega : \Omega^k(P, \mathfrak{g}) \rightarrow \Omega^{k+1}(P, \mathfrak{g})$$

defined by

$$d_\omega \eta(X_1, \dots, X_{k+1}) = d\eta(X_1^H, \dots, X_{k+1}^H),$$

where X_i^H are the projections of the vectors X_i onto the horizontal distribution

$$H = \ker \omega = \prod_{p \in P} \ker \omega_p.$$

Note that the exterior covariant derivative of $d_\omega \eta$ of any form $\eta \in \Omega^k(P, \mathfrak{g})$ is horizontal, cf. (3.1). Furthermore, if η is of Ad-type, then so is $d_\omega \eta$. If η is horizontal and of Ad-type, we have the *generalized Cartan identity*

$$d_\omega \eta = d\omega + [\omega \wedge \eta], \tag{3.2}$$

which is useful for computations.

The *curvature* of the connection ω is the 2-form

$$F_\omega = d_\omega \omega \in \Omega^2(P, \mathfrak{g}).$$

The curvature is a horizontal form of Ad-type on P . It satisfies the structure identity¹

$$F_\omega = d\omega + \frac{1}{2}[\omega \wedge \omega], \tag{3.3}$$

¹Not to be confused with the generalized Cartan identity (3.2). Note that ω is not necessarily horizontal, so the latter does not apply, which is also hinted by the factor $1/2$ in front of the commutator term.

as well as the Bianchi identity

$$d_\omega F_\omega = 0.$$

Under a gauge transformation $\sigma \mapsto \sigma g$, the pullback $\sigma^* F_\omega = F_\omega^\sigma$ transforms as

$$F_\omega^\sigma \mapsto \text{Ad}_{g^{-1}} \circ F_\omega^\sigma. \quad (3.4)$$

The connection and curvature are thus objects that naturally live on the total space P . However, in most cases one ultimately wants to work on the base manifold. There is a construction that allows this.

Let $G \rightarrow P \xrightarrow{\pi} M$ be a principal bundle. Let F be a smooth manifold and let H be a Lie group acting on F effectively to the left. Further let $\lambda : G \rightarrow H$ be a Lie group homomorphism. The *associated fiber bundle* is the fiber bundle $P \times_\lambda F \xrightarrow{\tilde{\pi}} M$ defined as the product $P \times F$ factored by the equivalence relation

$$(p, f) \sim (pg, \lambda(g^{-1})f), \quad g \in G,$$

and equipped with the natural projection $\tilde{\pi}([p, f]) = \pi(p)$.

As a particularly important example of this construction, consider the automorphism group $\text{Aut}(\mathfrak{g})$ which naturally acts on the Lie algebra $\mathfrak{g}$ to the left. The fiber bundle

$$\text{Ad } P := P \times_{\text{Ad}} \mathfrak{g} \xrightarrow{\tilde{\pi}} M$$

associated to the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ is called the *adjoint bundle*.

The point of introducing this construction is that the space of horizontal forms $\Omega^k(P, \mathfrak{g})$ of Ad-type on P is isomorphic to the space of forms $\Omega^k(M, \text{Ad } P)$ defined on M . Explicitly, given a horizontal $\tilde{\omega} \in \Omega^k(P, \mathfrak{g})$ of Ad-type, the corresponding $\omega \in \Omega^k(M, \text{Ad } P)$ is defined by

$$\omega_x(X_1, \dots, X_k) = [p, \tilde{\omega}_p(\tilde{X}_1, \dots, \tilde{X}_k)],$$

where $p \in P$ and $\tilde{X}_i \in T_p P$ satisfy $\pi(p) = x \in M$ and $d\pi_p(\tilde{X}_i) = X_i \in T_x M$. This definition is independent of the choice of vector fields X_i by the horizontality of $\tilde{\omega}$, and it is independent of the choice of p by the fact that $\tilde{\omega}$ is of Ad-type. The exterior covariant derivative can therefore also be viewed as a map

$$d_\omega : \Omega^k(M, \text{Ad } P) \rightarrow \Omega^{k+1}(M, \text{Ad } P).$$

Even though a connection form $\omega \in \Omega^1(P, \mathfrak{g})$ is not horizontal, we see that the difference of two connection forms is horizontal and of Ad-type, and can hence be identified with an element of $\Omega^1(M, \text{Ad } P)$. More importantly, the curvature form $F_\omega \in \Omega^2(P, \mathfrak{g})$ is both horizontal and of Ad-type, and can thus be identified with an element of $\Omega^2(M, \text{Ad } P)$.

Assume now that G is compact, so that we can find a bi-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on its Lie algebra $\mathfrak{g}$. If G is semi-simple, we may take define the metric to be a negative multiple of the Killing form

$$(X, Y) \mapsto \text{tr}(\text{ad}_X \circ \text{ad}_Y),$$

where ad is the adjoint endomorphism $\text{ad}_X Y = [X, Y]$. We can then define an inner product on $\text{Ad } P$ fiberwise by defining

$$\langle [p, X], [q, Y] \rangle_x = \langle X, Y \rangle_{\mathfrak{g}}.$$

Observe that this is well-defined and independent of chosen representatives by the assumed bi-invariance of the inner product on $\mathfrak{g}$. Thus we also obtain an inner product on k -forms $\Omega^k(M, \text{Ad } P)$. In particular,

$$\langle F_\omega, F_\omega \rangle_{\Omega^2(M, \text{Ad } P)} = \|F_\omega\|_{\Omega^2(M, \text{Ad } P)}^2 = \langle F_\omega \wedge \star F_\omega \rangle$$

defines a smooth function on M . By construction, for a given gauge $\sigma : U \subset M \rightarrow P$, we also have

$$\|\sigma^* F_\omega\|_{\Omega^2(M)}^2 = \|F_\omega\|_{\Omega^2(M, \text{Ad } P)}^2 \quad \text{in } U.$$

In a gauge, we can get an explicit formula that is useful for calculations. Indeed, if²

$$\sigma^* F_\omega = \frac{1}{2} dx^\mu \wedge dx^\nu \otimes F_{\mu\nu}, \quad F_{\mu\nu} \in \mathfrak{g},$$

we see that

$$\|\sigma^* F_\omega\|_{\Omega^2(M)}^2 = \frac{1}{4} \langle dx^\mu \wedge dx^\nu, dx^\alpha \wedge dx^\beta \rangle_g \langle F_{\mu\nu}, F_{\alpha\beta} \rangle = \frac{1}{2} \langle F_{\mu\nu}, F^{\mu\nu} \rangle.$$

Throughout the rest of this work, we will abuse the notation, and often simply denote this norm by $\|F_\omega\|$, regardless of which interpretation of F_ω we choose.

Having transferred the problem to the base manifold M , we can now define the *Yang-Mills functional* as

$$\Omega^1(P, \mathfrak{g}) \ni \omega \mapsto \int_M \|F_\omega\|^2 \text{vol}_g = \int_M \langle F_\omega \wedge \star F_\omega \rangle.$$

Critical points of this functional are called *Yang-Mills connections* and are of great importance in particle physics. In this work the Yang-Mills functional will appear as a part of the more general haunted Einstein-Yang-Mills theory, so we will not study it in the present section in more detail. Instead, we end this section by noting an integration by parts formula for the exterior covariant derivative which will turn out to be useful for calculating the variation of the functional. Namely, if $\omega \in \Omega^1(P, \mathfrak{g})$ is a connection, we have

$$\int_M \langle d_\omega \eta \wedge \star \theta \rangle = (-1)^{n(k-1)+p+1} \int_M \langle \eta \wedge \star d_\omega \star \theta \rangle, \quad (3.5)$$

where $\eta \in \Omega^k(M, \text{Ad } P)$ is compactly supported, $\theta \in \Omega^{k+1}(M, \text{Ad } P)$, and p is the index of the metric g .

²The factor $1/2$ seems to be standard in electromagnetism literature. I suppose this is because it makes it easier to read off the components of a given form, so e.g. with this notation, the form $F_\omega = Q dx^1 \wedge dx^2$ would have $F_{12} = -F_{21} = Q$.

3.2 Spherically symmetric $\mathbf{SU}(2)$ connections

Throughout the rest of this work, we will focus on the case when the gauge group $G = \mathbf{SU}(2)$. We therefore wish to understand principal $\mathbf{SU}(2)$ -bundles over a spherically symmetric spacetime manifold $M = \mathbb{R}^2 \times \mathbb{S}^2$ (cf. Definition 2.1.1), as well as connections on these bundles. To do this, we will impose additional symmetry, i.e. we will also introduce the notion of spherical symmetry for principal bundles and connections. We will then show (see Theorem 3.2.5 for a more precise statement) that the most general non-trivial spherically symmetric connection on a spherically symmetric $\mathbf{SU}(2)$ -bundle has the well-known form (in a gauge)

$$\begin{aligned} \omega = & (w \, d\theta - v \sin \theta \, d\varphi) \otimes X + (v \, d\theta + w \sin \theta \, d\varphi) \otimes Y \\ & + (a \, dt + b \, d\rho + \cos \theta \, d\varphi) \otimes Z, \end{aligned}$$

where

- ◇ $(t, \rho, \theta, \varphi) \in \mathbb{R}^2 \times \mathbb{S}^2$ are the spacetime coordinates,
- ◇ w, v, a, b are smooth functions of (t, ρ) ,
- ◇ X, Y, Z is the standard basis for the Lie algebra $\mathfrak{su}(2)$ given by

$$X = -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = -\frac{i}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = -\frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (3.6)$$

satisfying the commutator relations

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

Although this ansatz for the connection form is well-known and has been used in what seems like an infinitude of research articles concerning $\mathbf{SU}(2)$ Yang-Mills theory, it seems to be difficult to find references where this is derived explicitly. We therefore present the derivation here in more detail, using a more general theory of invariant connections over principal bundles with sufficient symmetry. We will first state some general definitions and results, and then specialize to our particular setting.

Definition 3.2.1. Let M be a smooth manifold and K a Lie group (called the *symmetry group*). We say that M is *K-symmetric* if it admits a decomposition of the form

$$M \cong N \times K/H$$

for some smooth connected manifold N and closed Lie subgroup $H \subset K$.

Note that every K -symmetric manifold M has a natural left action of K . Indeed, since K acts on K/H by multiplying the cosets to the left, we may also define an action on M that leaves the N -factor unchanged.

Definition 3.2.2. Let $M = N \times K/H$ be a K -symmetric manifold, and let G be a compact Lie group (called the *structure/gauge group*).

- ◊ We say that the principal G -bundle $P \xrightarrow{\pi} M$ is K -symmetric if the action of K on M admits a lift to an effective left action on P by bundle automorphisms (i.e. the lifted action of K should commute with the bundle G -action).
- ◊ We say that two K -symmetric principal G -bundles $P \xrightarrow{\pi} M$ and $\tilde{P} \xrightarrow{\tilde{\pi}} M$ are *equivalent* if there exists a map $\Phi : P \rightarrow \tilde{P}$ such that $\tilde{\pi} \circ \Phi = \pi$ which is equivariant both with respect to the right action of G (so that P and $\tilde{P}$ are isomorphic as principal G -bundles) and with respect to the left action of K .
- ◊ A connection $\omega \in \Omega^1(P, \mathfrak{g})$ on a K -symmetric bundle P is said to be K -invariant if

$$\ell_k^* \omega = \omega, \quad \text{for all } k \in K,$$

where ℓ_k denotes the left action by $k \in K$ on P .

In both of the previous definitions, when $K = \mathbf{SU}(2)$, then the a K -symmetric manifold/bundle/connection is often also said to be *spherically symmetric*.

Having defined all the necessary notions, we can now state a general theorem which gives a characterization of K -symmetric bundles over a K -symmetric manifold, and also a characterization of the K -invariant connections on such bundles.

Theorem 3.2.3 ([Bro96; HSV08]). *Let $M = N \times K/H$ be a K -symmetric manifold and G a Lie group. Denote the Lie algebras of K, H, G respectively by $\mathfrak{k}, \mathfrak{h}, \mathfrak{g}$.*

- (i) *The equivalence classes of K -symmetric principal G -bundles over M are in one-to-one correspondence with conjugacy classes of Lie group homomorphisms³ $\lambda : H \rightarrow G$. More explicitly, for a homomorphism $\lambda : H \rightarrow G$, define the associated bundle $G \rightarrow P_\lambda \xrightarrow{\pi} M$ as the product $N \times K \times G$ factored by the equivalence relation*

$$(y, k, g) \sim (y, kh, \lambda(h^{-1})g), \quad h \in H,$$

where the projection and the actions of $k_0 \in K$ and $g_0 \in G$ are defined respectively as

$$\pi([y, k, g]) = (y, [k]), \quad k_0[y, k, g] = [y, k_0 k, g], \quad [y, k, g]g_0 = [y, k, gg_0],$$

for $[y, k, g] \in P_\lambda$. Then P_λ is a K -symmetric principal G -bundle over M . Conversely, any K -symmetric principal G -bundle P over M is isomorphic to P_λ as defined above, where $\lambda = \lambda_{p_0}$ is the isotropy homomorphism defined by the relation

$$hp_0 = p_0 \lambda(h), \quad h \in H,$$

for some given fixed $p_0 \in P$.

³Two homomorphisms $\lambda, \mu : H \rightarrow G$ are said to be *conjugate* if there exists a $g \in G$ such that

$$\lambda(h) = g^{-1} \mu(h) g, \quad h \in H.$$

(ii) The K -invariant connections on the bundle P_λ are uniquely determined by:

◇ A smooth map

$$\Lambda : N \rightarrow W_\lambda,$$

where W_λ is the space of linear maps $\Lambda : \mathfrak{k} \rightarrow \mathfrak{g}$ satisfying the Wang conditions

$$\diamond \Lambda \circ \text{Ad}_h = \text{Ad}_{\lambda(h)} \circ \Lambda \text{ for all } h \in H,$$

$$\diamond \Lambda|_{\mathfrak{h}} = d\lambda.$$

◇ A one-form η on N with values in the subalgebra of $\mathfrak{h}$ which is invariant under the adjoint action of $\lambda(H)$.

Explicitly, a choice of section $\sigma : U \rightarrow K$ of the bundle $K \rightarrow K/H$ induces a gauge on P_λ under which the K -invariant connection corresponding to (Λ, η) pulls down to the gauge potential

$$\omega = \Lambda \circ \sigma^* \mu^K + \eta \quad (3.7)$$

on $N \times U \subset M$, where μ^K is the Maurer-Cartan form on K .

In the context of the present manuscript, we will work with gauge group $G = \mathbf{SU}(2)$, and we are interested in spherically symmetric manifolds of the form

$$M = \mathbb{R}^2 \times \mathbb{S}^2.$$

In terms of the general notation employed above, we thus have

$$N = \mathbb{R}^2, \quad K = \mathbf{SU}(2) \cong \mathbb{S}^3, \quad H = \mathbf{U}(1) \cong \mathbb{S}^1.$$

We will work with the already mentioned basis (3.6) for $\mathfrak{su}(2)$. In particular, $\mathfrak{u}(1) \subset \mathfrak{su}(2)$ is spanned by Z , and we can view $\mathbf{U}(1)$ as the subgroup $\exp(\mathbb{R}Z)$ of $\mathbf{SU}(2)$. Here, the homogeneous space

$$K/H = \mathbf{SU}(2)/\mathbf{U}(1) \cong \mathbb{S}^2$$

can be viewed as a $\mathbf{U}(1)$ -bundle via the Hopf fibration⁴

$$\pi : \mathbb{S}^3 \subset \mathbb{C}^2 \rightarrow \mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R}, \quad (z, w) \mapsto (2i\bar{z}w, |z|^2 - |w|^2), \quad (3.8)$$

where we employ the standard identification

$$\mathbf{SU}(2) \ni \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \cong (z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2, \quad |z|^2 + |w|^2 = 1.$$

Lemma 3.2.4. *The conjugacy classes of homomorphisms $\lambda : \mathbf{U}(1) \rightarrow \mathbf{SU}(2)$ are given by the non-negative integral powers*

$$\lambda = \lambda_n : \exp(sZ) \mapsto \exp(snZ), \quad 0 \leq n \in \mathbb{Z}.$$

⁴For later convenience, we modify the first factor of the fibration to be $2i\bar{z}w$ (more commonly this is taken to be $2z\bar{w}$). This can also be viewed as a modification of the identification $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$.

Proof. Firstly, λ_n is clearly a homomorphism for any $n \in \mathbb{Z}$. Conversely, we need to show that any other homomorphism $\mu : \mathbf{U}(1) \rightarrow \mathbf{SU}(2)$ is conjugate to some λ_n for $n \geq 0$. To see this, note that the elements of the image $\mu(\mathbf{U}(1))$ are unitary matrices which commute (since μ is a homomorphism and $\mathbf{U}(1)$ is abelian), and therefore they admit a simultaneous diagonalization by a unitary matrix $g \in \mathbf{U}(2)$, in view of the spectral theorem. Since the elements of $\mu(\mathbf{U}(1)) \subset \mathbf{SU}(2)$ furthermore have unit determinant, the matrix g can be chosen in $\mathbf{SU}(2)$. Therefore, μ is given by

$$\mu : \exp(sZ) \mapsto g^{-1} \exp(\alpha(s)Z)g \quad (3.9)$$

for some $\alpha : \mathbb{R} \rightarrow \mathbb{R}$. Then μ is a homomorphism if and only if $\tilde{\mu} : \exp(sZ) \mapsto \exp(\alpha(s)Z)$ is an endomorphism of $\mathbf{U}(1)$. But it is well-known that this is the case if and only if $\alpha(s) = ns$ for some $n \in \mathbb{Z}$.⁵ For the case of μ , the integer n can furthermore be assumed to be non-negative, since an additional conjugation in (3.9) e.g. by $\exp(\pi Y) \in \mathbf{SU}(2)$ sends $\alpha \mapsto -\alpha$. $\square$

Remark. The results of [HSV08] also show that the bundle $P_\lambda \rightarrow M$ associated to the homomorphism $\lambda : H \rightarrow G$ is trivial (i.e. admits a global section and is thus isomorphic to the trivial principal G -bundle over M) if and only if λ can be extended to a smooth map $\tilde{\lambda} : K \rightarrow G$ such that

$$\tilde{\lambda}(kh) = \tilde{\lambda}(k)\lambda(h), \quad k \in K, h \in H.$$

In the context of Lemma 3.2.4, this is the case if and only if $n = 0$ or $n = 1$.

We continue by studying the K -invariant connections induced by the homomorphisms λ_n . Note that the differential $d\lambda_n : \mathfrak{u}(1) \rightarrow \mathfrak{su}(2)$ satisfies

$$d\lambda_n(Z) = \left. \frac{d}{ds} \right|_{s=0} \lambda_n \circ \exp(sZ) = \left. \frac{d}{ds} \right|_{s=0} \exp(snZ) = nZ.$$

Thus, if Λ_n is a map satisfying the Wang conditions corresponding to λ_n , cf. Theorem 3.2.3 (ii), we get

$$\Lambda_n(Z) = d\lambda_n(Z) = nZ.$$

On the other hand,

$$\begin{aligned} \Lambda_n(X) &= -\Lambda_n([Z, Y]) \\ &= -\left. \frac{d}{ds} \right|_{s=0} \Lambda_n \circ \text{Ad}_{\exp(sZ)}(Y) \\ &= -\left. \frac{d}{ds} \right|_{s=0} \text{Ad}_{\exp(snZ)} \circ \Lambda_n(Y) \\ &= -n[Z, \Lambda_n(Y)]. \end{aligned}$$

⁵This can be seen e.g. by calculating the differential $d\tilde{\mu}$ and using the fact that the endomorphisms of $\mathfrak{u}(1) \cong \mathbb{R}$ must be of the form $x \mapsto nx$ for some $n \in \mathbb{R}$. The constant n then must be an integer in order for the map $\tilde{\mu}$ to be well-defined.

In an entirely similar manner,

$$\Lambda_n(Y) = n[Z, \Lambda_n(X)].$$

Since $\Lambda_n(X), \Lambda_n(Y) \in \mathfrak{su}(2)$, we may write them in the basis (3.6) as

$$\Lambda_n(X) = wX + vY + uZ, \quad \Lambda_n(Y) = aX + bY + cZ,$$

so that the latter two conditions reduce to

$$u = c = w - nb = b - nw = v + na = a + nv = 0.$$

This system has non-trivial solutions if and only if $n = 1$. In particular, we get

$$\Lambda_n(X) = \begin{cases} 0, & n \neq 1, \\ wX + vY, & n = 1 \end{cases}, \quad \Lambda_n(Y) = \begin{cases} 0, & n \neq 1, \\ -vX + wY, & n = 1. \end{cases}$$

Next, we wish to calculate “spherical part” of the connection form (3.7), so we need to fix a gauge, i.e. a section of the bundle $\mathbf{SU}(2) \rightarrow \mathbf{SU}(2)/\mathbf{U}(1) \cong \mathbb{S}^2$. Consider the map $\sigma : (0, \pi) \times [0, 2\pi) \rightarrow \mathbf{SU}(2)$ given by

$$\sigma(\theta, \varphi) = \exp(\varphi Z) \exp(\theta X) = \begin{bmatrix} e^{-\frac{i\varphi}{2}} \cos \frac{\theta}{2} & -ie^{-\frac{i\varphi}{2}} \sin \frac{\theta}{2} \\ -ie^{\frac{i\varphi}{2}} \sin \frac{\theta}{2} & e^{\frac{i\varphi}{2}} \cos \frac{\theta}{2} \end{bmatrix}.$$

Note that under the Hopf fibration (3.8), we have

$$\sigma(\theta, \varphi) \cong \left(e^{-\frac{i\varphi}{2}} \cos \frac{\theta}{2}, -ie^{\frac{i\varphi}{2}} \sin \frac{\theta}{2} \right) \xrightarrow{\pi} (e^{i\varphi} \sin \theta, \cos \theta) \cong (\theta, \varphi) \in \mathbb{S}^2,$$

which shows that σ defines a section of the bundle $\mathbf{SU}(2) \rightarrow \mathbf{SU}(2)/\mathbf{U}(1)$ with respect to the standard coordinates (θ, φ) on $\mathbb{S}^2 \setminus \{\text{poles}\}$. Now

$$d\sigma = d\theta \otimes \sigma X + d\varphi \otimes Z\sigma,$$

so that the pullback of the Maurer-Cartan form $\mu = \mu^{\mathbf{SU}(2)}$ on $\mathbf{SU}(2)$ by σ is given by

$$\begin{aligned} \sigma^* \mu &= \sigma^{-1} d\sigma \\ &= d\theta \otimes X + d\varphi \otimes \sigma^{-1} Z\sigma \\ &= d\theta \otimes X + \sin \theta d\varphi \otimes Y + \cos \theta d\varphi \otimes Z. \end{aligned} \tag{3.10}$$

We therefore get

$$\begin{aligned} \Lambda_n \circ \sigma^* \mu &= d\theta \otimes \Lambda(X) + \sin \theta d\varphi \otimes \Lambda(Y) + \cos \theta d\varphi \otimes \Lambda(Z) \\ &= (w d\theta - v \sin \theta d\varphi) \otimes X + (v d\theta + w \sin \theta d\varphi) \otimes Y + n \cos \theta d\varphi \otimes Z, \end{aligned}$$

where w, v are smooth functions on $N = \mathbb{R}^2$, and are assumed to be zero unless $n = 1$.

To complete the connection form (3.7), we still need a one-form on $N = \mathbb{R}^2$ with values in the subalgebra of elements of $\mathfrak{u}(1)$ which are invariant under the adjoint action of $\lambda(\mathbf{U}(1)) = \mathbf{U}(1)$. But this subalgebra is the entire $\mathfrak{u}(1)$, so that any form η which is a multiple of Z is eligible, i.e.

$$\eta = (a \, dt + b \, d\rho) \otimes Z,$$

where (t, ρ) denote the coordinates on $\mathbb{R}^2$ and a, b are smooth functions of (t, ρ) . Invoking now Theorem 3.2.3, we obtain the following.

Theorem 3.2.5. *The equivalence classes of $\mathbf{SU}(2)$ -symmetric principal $\mathbf{SU}(2)$ -bundles over $M = \mathbb{R}^2 \times \mathbb{S}^2$ are in one-to-one correspondence with non-negative integers. If ω is an $\mathbf{SU}(2)$ -invariant connection over the bundle corresponding to $0 \leq n \in \mathbb{Z}$, then a gauge can be constructed so that the connection form pulls down to the one-form*

$$\begin{aligned} \omega = & (w \, d\theta - v \sin \theta \, d\varphi) \otimes X + (v \, d\theta + w \sin \theta \, d\varphi) \otimes Y \\ & + (a \, dt + b \, d\rho + n \cos \theta \, d\varphi) \otimes Z, \end{aligned}$$

on M , where w, v, a, b are smooth functions of $(t, \rho) \in \mathbb{R}^2$, and (θ, φ) are the standard coordinates on $\mathbb{S}^2$. Furthermore, w and v are identically zero unless $n = 1$.

Remark 3.2.6. We note that this form of ω is preserved by $\mathbf{U}(1)$ gauge transformations of the form $h = \exp(-\alpha Z)$, where α is a smooth function of (t, ρ) . Indeed, we have

$$\begin{aligned} h^{-1} X h &= \cos(\alpha) X + \sin(\alpha) Y, \\ h^{-1} Y h &= -\sin(\alpha) X + \cos(\alpha) Y, \\ h^{-1} Z h &= Z, \\ h^{-1} dh &= -(\partial_t \alpha \, dt + \partial_\rho \alpha \, d\rho) \otimes Z \end{aligned}$$

so that the transformation law

$$\omega \mapsto h^{-1} \omega h + h^{-1} dh$$

therefore gives

$$\begin{aligned} w &\mapsto w \cos \alpha - v \sin \alpha, & a &\mapsto a - \partial_t \alpha, \\ v &\mapsto w \sin \alpha + v \cos \alpha, & b &\mapsto b - \partial_\rho \alpha. \end{aligned} \tag{3.11}$$

We now make a few simplifying remarks for the static case, i.e. when the functions w, v, a, b describing the connection depend only on ρ :

- (i) We can make a $\mathbf{U}(1)$ gauge transformation depending only on ρ so as to achieve $b \equiv 0$. Explicitly, we should select

$$\alpha(\rho) = \alpha_0 + \int_0^\rho b$$

for some $\alpha_0 \in \mathbb{R}$. The gauge thus obtained is sometimes called the *radial gauge*.

- (ii) Observe that the same cannot be done to achieve $a \equiv 0$, as that would require a time-dependent gauge transformation (i.e. one depending on t), which would also make the connection time-dependent.
- (iii) If v happens to be a constant multiple of w , then we can make a constant $\mathbf{U}(1)$ gauge transformation so as to make $v \equiv 0$. Explicitly, if $v = \beta w$, we can take

$$\alpha_0 = -\arctan \beta,$$

cf. (i). It will in fact turn out from the Yang-Mills equations that this is indeed the case.

Let us hereafter assume $n = 1$ and that the local gauge potential is static and given in the radial gauge as described above, i.e.

$$\omega = (w d\theta - v \sin \theta d\varphi) \otimes X + (v d\theta + w \sin \theta d\varphi) \otimes Y + (a dt + \cos \theta d\varphi) \otimes Z,$$

where w, v, a are functions only of ρ . Let us compute the curvature using the structure identity (3.3). We have

$$\begin{aligned} d\omega &= (\dot{w} d\rho \wedge d\theta - \dot{v} d\rho \wedge \sin \theta d\varphi - v \cos \theta d\theta \wedge d\varphi) \otimes X \\ &\quad + (\dot{v} d\rho \wedge d\theta + \dot{w} d\rho \wedge \sin \theta d\varphi + w \cos \theta d\theta \wedge d\varphi) \otimes Y \\ &\quad - (\dot{a} dt \wedge d\rho + d\theta \wedge \sin \theta d\varphi) \otimes Z, \\ \frac{1}{2}[\omega \wedge \omega] &= -(av dt \wedge d\theta + aw dt \wedge \sin \theta d\varphi - v \cos \theta d\theta \wedge d\varphi) \otimes X \\ &\quad + (aw dt \wedge d\theta - av dt \wedge \sin \theta d\varphi - w \cos \theta d\theta \wedge d\varphi) \otimes Y \\ &\quad + (w^2 + v^2) d\theta \wedge \sin \theta d\varphi \otimes Z. \end{aligned}$$

The curvature of ω therefore satisfies

$$\begin{aligned} F_\omega &= d\omega + \frac{1}{2}[\omega \wedge \omega] \\ &= -(av dt - \dot{w} d\rho) \wedge (d\theta \otimes X + \sin \theta d\varphi \otimes Y) \\ &\quad + (aw dt + \dot{v} d\rho) \wedge (d\theta \otimes Y - \sin \theta d\varphi \otimes X) \\ &\quad - (\dot{a} dt \wedge d\rho + (1 - w^2 - v^2) d\theta \wedge \sin \theta d\varphi) \otimes Z. \end{aligned} \tag{3.12}$$

In particular, see that the connection is flat if $a \equiv 0$ and $w^2 + v^2 \equiv 1$ with both w and v constant. This is unsurprising since in this case we can make $v \equiv 0$ by a constant gauge change (cf. Remark 3.2.6) above), and for $w \equiv 1$ the gauge potential ω is just the pullback of the Maurer-Cartan form on $\mathbf{SU}(2)$, cf. 3.10.

We would like to end this chapter with a discussion of the notion of charge in Yang-Mills theory. In analogy with Maxwell's theory of electromagnetism, one would ideally like to define these as the Lie algebra valued limits

$$\lim_{\rho \rightarrow \pm\infty} \oint_{\mathbb{S}_\rho^2} \star F_\omega \quad \text{and} \quad \lim_{\rho \rightarrow \pm\infty} \oint_{\mathbb{S}_\rho^2} F_\omega$$

where $\mathbb{S}_\rho^2$ denotes the spatial 2-sphere of radius $r(\rho)$. Note that the resulting quantities are gauge-dependent, cf. [CK87] for a detailed discussion. If the Lie algebra also comes equipped with a bi-invariant metric, then one can define the Yang-Mills charges as the scalars

$$Q^\pm = \lim_{\rho \rightarrow \pm\infty} \left\| \frac{1}{8\pi} \oint_{\mathbb{S}_\rho^2} \star F_\omega \right\| \quad \text{and} \quad P^\pm = \lim_{\rho \rightarrow \pm\infty} \left\| \frac{1}{8\pi} \oint_{\mathbb{S}_\rho^2} F_\omega \right\|,$$

cf. [BP92]. In analogy with electromagnetism, we refer to $Q^\pm$ and $P^\pm$ respectively as the asymptotic *electric* and *magnetic Yang-Mills charges*. For the connection ω with curvature (3.12), these are given by

$$Q^\pm = \lim_{\rho \rightarrow \pm\infty} r e^{-\tau} |\dot{a}|, \quad P^\pm = \lim_{\rho \rightarrow \pm\infty} |1 - w^2 - v^2|. \quad (3.13)$$

We will search for so-called *hairy* solutions.⁶ These have vanishing charges at infinity $Q^\pm = P^\pm = 0$, but the potentials a, w, v are not all trivial. In particular, we will later make the simplifying assumptions $a \equiv v \equiv 0$, so that hairy solutions will have non-constant w and satisfy the boundary condition $|w| \rightarrow 1$.

⁶The nomenclature originates from Wheeler, who famously said that *black holes have no hair*. More precisely, with this statement he wished to capture the celebrated static no-hair theorem for the Einstein-Maxwell theory, which roughly says that the static asymptotically flat black hole solutions of the Einstein-Maxwell equations are completely determined by the mass and the electric/magnetic charge, and are in particular described by the Reissner-Nordström family. One of the big surprises of replacing the Maxwell theory with more general Yang-Mills theories was the disproval of a straightforward generalization of such a no-hair theorem. Nevertheless, one can still obtain theorems of “no-hair type”, an example of which we will see in §4.5.

Chapter 4

Haunted Einstein-Yang-Mills theory

In this chapter, we will describe the Einstein-Yang-Mills theory coupled to a phantom field. The theory can be viewed as a local Lagrangian theory where the Lagrangian is composed of the Einstein-Hilbert Lagrangian, the Yang-Mills Lagrangian, and a Lagrangian for a phantom scalar field. Even though the theory behind each of these Lagrangians separately is well-studied in literature (see e.g. [Cho15; Ble81]), it would appear that references for the combination that we study in this manuscript are somewhat difficult to find, mainly due to the phantom nature of the scalar field. In fact, the articles [Shc12; Hau+14] are some of the only ones the author is aware of. We will therefore present the theory in a mathematically rigorous manner here, including all the relevant calculational details. We will also derive the Euler-Lagrange equations for the static spherically symmetric setting with gauge group $\mathbf{SU}(2)$, study some classical solutions within the framework of this theory, and also prove a “no-hair” theorem.

4.1 Lagrangian and Euler-Lagrange equations

Let M be a smooth n -dimensional manifold, G a compact Lie group with a bi-invariant metric on its Lie algebra $\mathfrak{g}$, and $G \rightarrow P \xrightarrow{\pi} M$ a principal G -bundle over M . We define:

- ◊ $\text{Met}(M)$ as the set of semi-Riemannian metrics on M with a given fixed index,¹
- ◊ $\text{Con}(P)$ as the set of connections on the principal bundle $G \rightarrow P \xrightarrow{\pi} M$,
- ◊ $C^k(M)$ as the set of k -times continuously differentiable real-valued functions on M .

We define the product space

$$X := \text{Met}(M) \times \text{Con}(P) \times C^2(M),$$

¹For our purposes it suffices to take $n = 4$ and consider only Lorentzian metrics on M , i.e. those with index equal to 1, but the formulation of the theory remains unchanged in the more general case.

and we consider the local Lagrangian theory induced by the map $L : X \rightarrow \Omega^n(M)$ given by

$$L(g, \omega, \phi) = (R_g - \|F_\omega\|^2 + \|\mathrm{d}\phi\|^2) \mathrm{vol}_g. \quad (4.1)$$

Alternatively, one can also write the Lagrangian using the language of forms (resp. in coordinates/gauge) as

$$\begin{aligned} L(g, \omega, \phi) &= R_g \mathrm{vol}_g - \langle F_\omega \wedge \star F_\omega \rangle + \mathrm{d}\phi \wedge \star \mathrm{d}\phi \\ &= \left(R_g - \frac{1}{2} \langle F_{\mu\nu}, F^{\mu\nu} \rangle + \partial_\mu \phi \partial^\mu \phi \right) \sqrt{|\det g|} \, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n, \end{aligned}$$

where $F_{\mu\nu}$ are the components of the curvature F_ω in a local gauge, i.e. $F_\omega = \frac{1}{2} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes F_{\mu\nu}$. As one usually does, we employ the principle of least action and look for stationary points, in the following sense.

Definition 4.1.1. A triplet $(g, \omega, \phi) \in X$ is said to be a *haunted Einstein-Yang-Mills system* if for any compact $K \subset M$ and variation

$$(g(t), \omega(t), \phi(t)) = (g, \omega, \phi) + t(h, \eta, \psi) \in X$$

such that $\mathrm{supp}_M(h, \eta, \psi) \subset K$, we have

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \int_K L(g(t), \omega(t), \phi(t)) = 0.$$

Here, $\mathrm{supp}_M(h, \eta, \psi)$ denotes the union of the projected supports in M , i.e. of

$$\begin{aligned} \mathrm{supp}_M(h) &= \mathrm{cl} \{x \in M \mid h_x(X, Y) \neq 0 \text{ for some } X, Y \in T_x M\} \\ \mathrm{supp}_M(\eta) &= \mathrm{cl} \{x \in M \mid \eta_p(X, Y) \neq 0 \text{ for some } p \in \pi^{-1}(x) \text{ and } X, Y \in T_p P\} \\ \mathrm{supp}_M(\psi) &= \mathrm{cl}(\psi^{-1}(0)), \end{aligned}$$

with cl denoting closure. In fact, since the difference of two connections is horizontal, we could also view η as an element of $\Omega^1(M, \mathrm{Ad} P)$ and extract its support there, but this results in the same set of course.

At this point the reader might be wondering about the (lack of) physical constants in front of the terms in the definition of the Lagrangian (4.1). In fact, if we consider

$$L^{\alpha, \beta, \gamma}(g, \omega, \phi) = (\alpha R_g - \beta \|F_\omega\|^2 + \gamma \|\mathrm{d}\phi\|^2) \mathrm{vol}_g,$$

for positive (real) constants α, β, γ , then we see that

$$L^{\alpha, \beta, \gamma} \left(\frac{\alpha}{\beta} g, \omega, \sqrt{\frac{\alpha}{\gamma}} \phi \right) = \beta \left(\frac{\alpha}{\beta} \right)^{\frac{n}{2}} L(g, \omega, \phi),$$

where we use the identities

$$\mathrm{vol}_{cg} = c^{\frac{n}{2}} \mathrm{vol}_g, \quad R_{cg} = \frac{1}{c} R_g, \quad \|F_\omega\|_{cg}^2 = \frac{1}{c^2} \|F_\omega\|_g^2, \quad \|\mathrm{d}\phi\|_{cg}^2 = \frac{1}{c} \|\mathrm{d}\phi\|_g^2,$$

for any positive real constant c . Thus, one does not lose generality by making the simplifying assumption $\alpha = \beta = \gamma = 1$.

We continue by deriving the Euler-Lagrange equations associated with the problem in Definition 4.1.1.

Theorem 4.1.2. *A triplet $(g, \omega, \phi) \in X$ is a haunted Einstein-Yang-Mills system if and only if*

$$\begin{cases} \text{Ric}_g - 2\langle F_\omega \otimes F_\omega \rangle + \frac{1}{n-2}\|F_\omega\|^2 g + d\phi \otimes d\phi = 0, & (4.2a) \\ d_\omega \star F_\omega = 0, & (4.2b) \\ \square_g \phi = 0, & (4.2c) \end{cases}$$

where

- ◇ n is the dimension of the base manifold M ,
- ◇ d_ω denotes the exterior covariant derivative with respect to the connection ω ,
- ◇ $\star$ is the Hodge star operator with respect to g ,
- ◇ $\square_g = g^{\mu\nu} \nabla_\mu \partial_\nu$ is the wave operator with respect to g ,
- ◇ the quantities related to F_ω are defined in a gauge and local coordinates on M by

$$\langle F_\omega \otimes F_\omega \rangle = \frac{1}{2} \langle F_{\mu\alpha}, F_\nu{}^\alpha \rangle dx^\mu \otimes dx^\nu, \quad \|F_\omega\|^2 = \frac{1}{2} \langle F_{\mu\nu}, F^{\mu\nu} \rangle,$$

where $F_\omega = \frac{1}{2} dx^\mu \wedge dx^\nu \otimes F_{\mu\nu}$. Note that $\text{tr}_g \langle F_\omega \otimes F_\omega \rangle = \|F_\omega\|^2$.

We refer to (4.2a) as the (trace-reversed) *Einstein field equation*, (4.2b) as the *Yang-Mills equation*, and (4.2c) as the *phantom field equation*.

Proof. As already mentioned at the start of the chapter, the variation of each of the terms in the Lagrangian is fairly standard in literature, but we present some details here to make sure we get the correct signs/constants between the terms in the final equations. For simplicity we denote

$$\begin{aligned} L_E(t) &= R_{g(t)} \text{vol}_{g(t)}, \\ L_{YM}(t) &= \|F_{\omega(t)}\|_{g(t)}^2 \text{vol}_{g(t)} \\ L_P(t) &= \|d\phi(t)\|_{g(t)}^2 \text{vol}_{g(t)}, \end{aligned}$$

so that $L = L_E - L_{YM} + L_P$. Following Definition 4.1.1, we consider a variation

$$(g(t), \omega(t), \phi(t)) = (g, \omega, \phi) + t(h, \theta, \psi), \quad t \in (-\varepsilon, \varepsilon),$$

where (h, η, ψ) has compact (projected) support in M . Throughout the proof, we will use the standard notation

$$\left. \frac{d}{dt} \right|_{t=0} =: \delta.$$

We first observe that a differentiation of the identity $g(t)^{\mu\alpha}g(t)_{\alpha\nu} = 1^\mu{}_\nu$ (the latter denoting the Kronecker delta) implies that

$$\delta g(t)^{\mu\nu} = -h^{\mu\nu},$$

and so in particular the variation of the trace operates as

$$\delta \operatorname{tr}_{g(t)} = -\operatorname{tr}_h + \operatorname{tr}_g \delta.$$

Let us calculate $\delta L_E(t)$. To start with, we have

$$\delta R_{g(t)} = \delta \operatorname{tr}_{g(t)} \operatorname{Ric}_{g(t)} = -\operatorname{tr}_h \operatorname{Ric}_g + \operatorname{tr}_g \delta \operatorname{Ric}_{g(t)},$$

so it suffices to compute the variation of the Ricci curvature. To this end, we recall that in local coordinates

$$\operatorname{Ric}_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\lambda\nu} + \Gamma^\lambda_{\lambda\sigma} \Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda},$$

where the Christoffel symbols satisfy

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

A calculation in normal coordinates (this is permitted since $\delta\Gamma$ is tensorial) shows that

$$(\delta\Gamma_{g(t)})^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\nabla_\mu h_{\nu\sigma} + \nabla_\nu h_{\mu\sigma} - \nabla_\sigma h_{\mu\nu})$$

$$(\delta\operatorname{Ric}_{g(t)})_{\mu\nu} = \nabla_\lambda (\delta\Gamma_{g(t)})^\lambda_{\mu\nu} - \nabla_\mu (\delta\Gamma_{g(t)})^\lambda_{\lambda\nu},$$

where ∇ is the covariant derivative with respect to g . Using these formulae, one may calculate

$$\begin{aligned} \operatorname{tr}_g \delta\operatorname{Ric}_{g(t)} &= \nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^\sigma \nabla_\sigma (g^{\mu\nu} h_{\mu\nu}) \\ &= \operatorname{div}_g \operatorname{div}_g h - \square_g \operatorname{tr}_g h = \operatorname{div}_g Y, \end{aligned}$$

where Y is the compactly supported vector field defined by

$$Y = \operatorname{div}_g h - \nabla \operatorname{tr}_g h.$$

It follows that

$$\delta R_{g(t)} = -\operatorname{tr}_h \operatorname{Ric}_g + \operatorname{div}_g Y.$$

Next, interpreting g and h as matrices, we have, by Jacobi's formula,

$$\delta \det g(t) = \operatorname{tr}(g^{-1}h) \det g = g^{\mu\nu} h_{\mu\nu} \det g = g_{\mu\nu} h^{\mu\nu} \det g = \operatorname{tr}_h(g) \det g,$$

and consequently,

$$\delta \operatorname{vol}_{g(t)} = \delta \sqrt{|\det g(t)|} dx = \frac{1}{2} \operatorname{tr}_h(g) \operatorname{vol}_g.$$

Hence,

$$\begin{aligned}\delta L_E(t) &= \delta R_{g(t)} \text{vol}_g + R_g \delta \text{vol}_{g(t)} \\ &= -\text{tr}_h \left(\text{Ric}_g - \frac{1}{2} R_g g \right) \text{vol}_g + (\text{div}_g Y) \text{vol}_g,\end{aligned}$$

and it follows that

$$\delta \int_K L_E(t) = - \int_K \text{tr}_h \left(\text{Ric}_g - \frac{1}{2} R_g g \right) \text{vol}_g,$$

where the last term vanishes by the divergence theorem.

Next, we consider the phantom Lagrangian L_P . We have

$$\begin{aligned}\delta \|\text{d}\phi(t)\|_{g(t)}^2 &= \delta \text{tr}_{g(t)}(\text{d}\phi(t) \otimes \text{d}\phi(t)) \\ &= -\text{tr}_h(\text{d}\phi \otimes \text{d}\phi) + 2\langle \text{d}\phi, \text{d}\psi \rangle_g\end{aligned}$$

Hence,

$$\begin{aligned}\delta L_P(t) &= \delta \|\text{d}\phi(t)\|_{g(t)}^2 \text{vol}_g + \|\text{d}\phi(t)\|_g^2 \delta \text{vol}_{g(t)} \\ &= -\text{tr}_h \left(\text{d}\phi \otimes \text{d}\phi - \frac{1}{2} \|\text{d}\phi\|_g^2 g \right) \text{vol}_g + 2\langle \text{d}\phi, \text{d}\psi \rangle_g \text{vol}_g,\end{aligned}$$

so that

$$\delta \int_K L_P(t) = - \int_K \text{tr}_h \left(\text{d}\phi \otimes \text{d}\phi - \frac{1}{2} \|\text{d}\phi\|_g^2 g \right) \text{vol}_g - 2 \int_K (\Box_g \phi) \psi \text{vol}_g.$$

where we use Green's identity on the last term (the boundary term vanishes because ψ has compact support).

Lastly, for the Yang-Mills term, we define the tensor field

$$\langle F_\omega \otimes F_\omega \rangle_g = \frac{1}{2} g^{\nu\beta} \langle F_{\mu\nu}, F_{\alpha\beta} \rangle \text{d}x^\mu \otimes \text{d}x^\alpha,$$

and calculate

$$\begin{aligned}\delta \|F_{\omega(t)}\|_{g(t)}^2 &= \delta \left(\frac{1}{2} g(t)^{\mu\alpha} g(t)^{\nu\beta} \langle (F_{\omega(t)})_{\mu\nu}, (F_{\omega(t)})_{\alpha\beta} \rangle \right) \\ &= -2 \text{tr}_h \langle F_\omega \otimes F_\omega \rangle_g + \delta \|F_{\omega(t)}\|_g^2.\end{aligned}$$

By the structure identity for the curvature we get

$$\begin{aligned}F_{\omega(t)} &= \text{d}\omega(t) + \frac{1}{2} [\omega(t) \wedge \omega(t)] \\ &= \text{d}\omega + t \text{d}\eta + \frac{1}{2} [\omega \wedge \omega] + t [\omega \wedge \eta] + \frac{t^2}{2} [\eta \wedge \eta] \\ &= F_\omega + t \text{d}\omega\eta + \frac{t^2}{2} [\eta \wedge \eta],\end{aligned}$$

where the last line follows from the generalized identity of Cartan (3.2). Therefore,

$$\delta \|F_{\omega(t)}\|_g^2 = 2\langle d_\omega \eta, F_\omega \rangle_g,$$

and we get

$$\begin{aligned} \delta L_{\text{YM}}(t) &= \delta \|F_{\omega(t)}\|^2 \text{vol}_g + \|F_\omega\|^2 \delta \text{vol}_{g(t)} \\ &= -\text{tr}_h \left(2\langle F_\omega \otimes F_\omega \rangle - \frac{1}{2} \|F_\omega\|^2 g \right) \text{vol}_g + 2\langle d_\omega \eta \wedge \star F_\omega \rangle_g. \end{aligned}$$

Now we note that by formula (3.5), we have

$$\int_K \langle d_\omega \eta \wedge \star F_\omega \rangle = (-1)^m \int_K \langle \eta \wedge \star d_\omega \star F_\omega \rangle,$$

for some natural number $m \in \mathbb{N}$ whose value is immaterial in this context. Hence,

$$\delta \int_K L_{\text{YM}}(t) = - \int_K \text{tr}_h \left(2\langle F_\omega \otimes F_\omega \rangle - \frac{1}{2} \|F_\omega\|^2 g \right) \text{vol}_g + 2(-1)^m \int_K \langle d_\omega \eta \wedge \star F_\omega \rangle$$

Summarizing the findings, we finally get

$$\begin{aligned} &\delta \int_K L(g(t), \omega(t), \phi(t)) \\ &= \delta \int_K L_E(t) - \delta \int_K L_{\text{YM}}(t) + \delta \int_K L_P(t) \\ &= 2(-1)^{m+1} \int_K \langle \eta \wedge \star d_\omega \star F_\omega \rangle - \int_K \square_g \phi \psi \text{vol}_g \\ &\quad - \int_K \text{tr}_h \left(\text{Ric}_g - 2\langle F_\omega \otimes F_\omega \rangle_g + d\phi \otimes d\phi - \frac{1}{2} (R_g - \|F_\omega\|_g^2 + \|d\phi\|_g^2) g \right) \text{vol}_g. \end{aligned}$$

Since the latter is supposed to vanish for all variations (h, η, ϕ) , we see that each term must be zero, and in particular we obtain the Einstein field equation

$$\text{Ric}_g - 2\langle F_\omega \otimes F_\omega \rangle_g + d\phi \otimes d\phi - \frac{1}{2} (R_g - \|F_\omega\|_g^2 + \|d\phi\|_g^2) g = 0,$$

the Yang-Mills equation (using also the fact that $\star$ is an isomorphism)

$$d_\omega \star F_\omega = 0,$$

and the phantom field equation

$$\square_g \phi = 0.$$

Taking the trace with respect to g in the Einstein field equation, we also see that

$$R_g - \|F_\omega\|_g^2 + \|d\phi\|_g^2 = \frac{2}{2-n} \|F_\omega\|_g^2,$$

which we may insert back into the Einstein field equation to simplify it. This yields an equivalent (so-called trace-reversed) formulation of the equation, as can be seen by taking the trace again and reinserting it into the equation, which again results in the original formulation. This completes the proof of Theorem 4.1.2. $\square$

4.2 Static spherically symmetric ansatz

We are now ready to reduce the system (4.2a–4.2c) of PDEs to a system of ODEs by making staticity (i.e. time independence) and spherical symmetry assumptions for the metric g , connection ω , and phantom field ϕ . In the first two subsections we compute the metric- and the connection-related quantities needed to write down the equations. A lot of these computations seem to be difficult to find in literature, so we present some more details here for completeness. The reader not interested in these details may skip to §4.2.3 without loss of continuity.

4.2.1 Metric

We consider the manifold $M = \mathbb{R}^2 \times \mathbb{S}^2$, equipped with the Lorentzian metric

$$g = -e^{2\tau(\rho)} dt \otimes dt + r(\rho)^2 (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

where $g_{\mathbb{S}^2}$ is the round metric on $\mathbb{S}^2$, given in standard spherical coordinates (θ, φ) by

$$g_{\mathbb{S}^2} = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi.$$

Throughout the rest of the manuscript, dot will denote differentiation with respect to ρ . The non-zero Christoffel symbols of the metric are given by

$$\begin{aligned} \Gamma_{t\rho}^t &= \dot{\tau}, & \Gamma_{tt}^\rho &= \frac{e^{2\tau}\dot{\tau}}{r^2}, & \Gamma_{\rho\rho}^\rho &= \frac{\dot{r}}{r}, \\ \Gamma_{\theta\theta}^\rho &= -\frac{\dot{r}}{r}, & \Gamma_{\varphi\varphi}^\rho &= -\frac{\dot{r}}{r} \sin^2 \theta, & \Gamma_{\rho\theta}^\theta &= \Gamma_{\theta\rho}^\theta = \frac{\dot{r}}{r}, \\ \Gamma_{\varphi\varphi}^\theta &= -\frac{1}{2} \sin(2\theta), & \Gamma_{\rho\varphi}^\varphi &= \Gamma_{\varphi\rho}^\varphi = \frac{\dot{r}}{r}, & \Gamma_{\theta\varphi}^\varphi &= \Gamma_{\varphi\theta}^\varphi = \cot \theta. \end{aligned}$$

Using this one can compute the non-zero components of the Ricci tensor

$$\begin{aligned} \text{Ric}_{tt} &= \frac{e^{2\tau}}{r^2} \left(\ddot{\tau} + \dot{\tau}^2 + \frac{\dot{r}\dot{\tau}}{r} \right), & \text{Ric}_{\theta\theta} &= 1 - \frac{\ddot{r}}{r} - \frac{\dot{r}\dot{\tau}}{r}, \\ \text{Ric}_{\rho\rho} &= - \left(\ddot{\tau} + \dot{\tau}^2 - \frac{\dot{r}\dot{\tau}}{r} - \frac{2\dot{r}^2}{r^2} + \frac{2\ddot{r}}{r} \right), & \text{Ric}_{\varphi\varphi} &= \text{Ric}_{\theta\theta} \sin^2 \theta. \end{aligned} \tag{4.3}$$

The volume form reads

$$\text{vol}_g = r^3 e^\tau dt \wedge d\rho \wedge d\theta \wedge \sin \theta d\varphi,$$

where $\sqrt{|\det g|} = r^3 e^\tau \sin \theta$. The wave operator is therefore given by

$$\begin{aligned} \square_g \phi &= \frac{1}{\sqrt{|\det g|}} \partial_\mu \left(g^{\mu\nu} \sqrt{|\det g|} \partial_\nu \phi \right) \\ &= \frac{1}{r^3 e^\tau} \frac{d}{d\rho} (r e^\tau \dot{\phi}) = \frac{1}{r^2} \left[\ddot{\phi} + \left(\dot{\tau} + \frac{\dot{r}}{r} \right) \dot{\phi} \right]. \end{aligned} \tag{4.4}$$

By the definition of the Hodge star operator, we can also calculate

$$\begin{aligned}
\star(dt \wedge d\rho) &= -re^{-\tau} d\theta \wedge \sin\theta d\varphi, & \star(dt \wedge \sin\theta d\varphi) &= -re^{-\tau} d\rho \wedge d\theta \\
\star(dt \wedge d\theta) &= re^{-\tau} d\rho \wedge \sin\theta d\varphi, & \star(d\rho \wedge \sin\theta d\varphi) &= \frac{e^\tau}{r} dt \wedge d\theta, \\
\star(d\rho \wedge d\theta) &= \frac{e^\tau}{r} dt \wedge \sin\theta d\varphi, & \star(d\theta \wedge \sin\theta d\varphi) &= \frac{e^\tau}{r} dt \wedge d\rho.
\end{aligned} \tag{4.5}$$

4.2.2 Connection

In Theorem 3.2.5, we have classified the $\mathbf{SU}(2)$ -invariant connections on $\mathbf{SU}(2)$ -symmetric principal bundles over M by an integer $n \in \mathbb{Z}$. For our ansatz we will consider the bundle corresponding to the non-trivial case $n = 1$. The most general $\mathbf{SU}(2)$ -invariant static connection ω over M can then be written as (cf. Theorem 3.2.5 and Remark 3.2.6) as

$$\omega = (w d\theta - v \sin\theta d\varphi) \otimes X + (v d\theta + w \sin\theta d\varphi) \otimes Y + (a dt + \cos\theta d\varphi) \otimes Z,$$

where $a, w, v \in C^2(\mathbb{R})$ are functions depending on ρ only, and the basis for the Lie algebra $\mathfrak{su}(2)$ is given by

$$X = -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = -\frac{i}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = -\frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These are normalized so as to satisfy the commutator relations

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

We also equip $\mathfrak{su}(2)$ with the inner product induced by its Killing form, i.e.

$$\langle A, B \rangle_{\mathfrak{su}(2)} = -\operatorname{tr}(\operatorname{ad}_A \circ \operatorname{ad}_B) = -4 \operatorname{tr}(AB), \quad A, B \in \mathfrak{su}(2).$$

The basis elements X, Y, Z are orthogonal with respect to this metric and²

$$\frac{1}{2} \|X\|_{\mathfrak{su}(2)}^2 = \frac{1}{2} \|Y\|_{\mathfrak{su}(2)}^2 = \frac{1}{2} \|Z\|_{\mathfrak{su}(2)}^2 = 1.$$

Recall from (3.12) that the curvature of ω satisfies

$$\begin{aligned}
F_\omega &= d\omega + \frac{1}{2}[\omega \wedge \omega] \\
&= -(av dt - \dot{w} d\rho) \wedge (d\theta \otimes X + \sin\theta d\varphi \otimes Y) \\
&\quad + (aw dt + \dot{v} d\rho) \wedge (d\theta \otimes Y - \sin\theta d\varphi \otimes X) \\
&\quad - (\dot{a} dt \wedge d\rho + (1 - w^2 - v^2) d\theta \wedge \sin\theta d\varphi) \otimes Z.
\end{aligned}$$

²We could have also chosen the metric $-2 \operatorname{tr}(AB)$ to make the basis elements orthonormal. However, the choice we made leads to nicer equations later and seems to be standard in literature. Ultimately it does not matter though, because changing this metric only amounts to modifying the corresponding term in the Lagrangian (4.1) by a constant factor, and we have seen that such constants can always be rescaled as we please.

In order to obtain the Yang-Mills equation (4.2b), we use (4.5) to calculate

$$\begin{aligned}\star F_\omega = & \left(re^{-\tau}aw \, d\rho + \frac{e^\tau \dot{v}}{r} \, dt \right) \wedge (d\theta \otimes X + \sin \theta \, d\varphi \otimes Y) \\ & + \left(re^{-\tau}av \, d\rho - \frac{e^\tau \dot{w}}{r} \, dt \right) \wedge (d\theta \otimes Y - \sin \theta \, d\varphi \otimes X) \\ & + \left(re^\tau \dot{a} \, d\theta \wedge \sin \theta \, d\varphi - \frac{e^\tau(1-w^2-v^2)}{r} \, dt \wedge d\rho \right) \otimes Z.\end{aligned}$$

We then easily see that

$$\begin{aligned}d \star F_\omega = & -\frac{d}{d\rho} \left(\frac{e^\tau \dot{v}}{r} \right) dt \wedge d\rho \wedge (d\theta \otimes X + \sin \theta \, d\varphi \otimes Y) \\ & - \cos \theta \left(re^{-\tau}aw \, d\rho + \frac{e^\tau \dot{v}}{r} \, dt \right) \wedge d\theta \wedge d\varphi \otimes Y \\ & + \frac{d}{d\rho} \left(\frac{e^\tau \dot{w}}{r} \right) dt \wedge d\rho \wedge (d\theta \otimes Y - \sin \theta \, d\varphi \otimes X) \\ & + \cos \theta \left(re^{-\tau}av \, d\rho - \frac{e^\tau \dot{w}}{r} \, dt \right) \wedge d\theta \wedge d\varphi \otimes X \\ & + \frac{d}{d\rho} (re^{-\tau} \dot{a}) \, d\rho \wedge d\theta \wedge \sin \theta \, d\varphi \otimes Z.\end{aligned}$$

On the other hand, an excruciating but straightforward calculation yields

$$\begin{aligned}[\omega \wedge \star F_\omega] = & -re^\tau v \left(e^{-2\tau}a^2 + \frac{1-w^2-v^2}{r^2} \right) dt \wedge d\rho \wedge (d\theta \otimes X + \sin \theta \, d\varphi \otimes Y) \\ & + re^\tau w \left(e^{-2\tau}a^2 + \frac{1-w^2-v^2}{r^2} \right) dt \wedge d\rho \wedge (d\theta \otimes Y - \sin \theta \, d\varphi \otimes X) \\ & - \cos \theta \left(re^{-\tau}av \, d\rho - \frac{e^\tau \dot{w}}{r} \, dt \right) \otimes X + \cos \theta \left(re^{-\tau}aw \, d\rho + \frac{e^\tau \dot{v}}{r} \, dt \right) \otimes Y \\ & + \left(\frac{2e^\tau(\dot{w}v - \dot{v}w)}{r} \, dt - 2re^{-\tau}a(w^2 + v^2) \, d\rho \right) \wedge d\theta \wedge \sin \theta \, d\varphi \otimes Z.\end{aligned}$$

We therefore finally end up with

$$\begin{aligned}
d_\omega \star F_\omega &= d \star F_\omega + [\omega \wedge \star F_\omega] \\
&= - \left[\frac{d}{d\rho} \left(\frac{e^\tau \dot{v}}{r} \right) + re^\tau v \left(e^{-2\tau} a^2 + \frac{1-w^2-v^2}{r^2} \right) \right] dt \wedge d\rho \wedge (d\theta \otimes X + \sin \theta d\varphi \otimes Y) \\
&\quad + \left[\frac{d}{d\rho} \left(\frac{e^\tau \dot{w}}{r} \right) + re^\tau w \left(e^{-2\tau} a^2 + \frac{1-w^2-v^2}{r^2} \right) \right] dt \wedge d\rho \wedge (d\theta \otimes Y - \sin \theta d\varphi \otimes X) \\
&\quad + \left[\frac{d}{d\rho} (re^{-\tau} \dot{a}) - 2re^{-\tau} q(w^2 + v^2) \right] d\rho \wedge d\theta \wedge \sin \theta d\varphi \otimes Z \\
&\quad + \frac{2e^\tau (\dot{w}v - \dot{v}w)}{r} dt \wedge d\theta \wedge \sin \theta d\varphi \otimes Z,
\end{aligned}$$

from which the Yang-Mills system can directly be read off. In particular, we see that $\dot{w}v - \dot{v}w = 0$, which implies that v is a constant multiple of w , and therefore we can assume $v \equiv 0$ without loss of generality by a making constant gauge change (cf. Remark 3.2.6). The remaining Yang-Mills equations are then

$$\begin{cases} \frac{d}{d\rho} \left(\frac{e^\tau \dot{w}}{r} \right) + re^\tau w \left(e^{-2\tau} a^2 + \frac{1-w^2}{r^2} \right) = 0, \\ \frac{d}{d\rho} (re^{-\tau} \dot{a}) - 2re^{-\tau} aw^2 = 0. \end{cases} \quad (4.6a)$$

$$\begin{cases} \frac{d}{d\rho} (re^{-\tau} \dot{a}) - 2re^{-\tau} aw^2 = 0. \end{cases} \quad (4.6b)$$

The formula for the curvature also simplifies considerably:

$$\begin{aligned}
F_\omega &= -\dot{a} dt \wedge d\rho \otimes Z + aw dt \wedge (d\theta \otimes Y - \sin \theta d\varphi \otimes X) \\
&\quad + \dot{w} d\rho \wedge (d\theta \otimes X + \sin \theta d\varphi \otimes Y) - (1-w^2) d\theta \wedge \sin \theta d\varphi \otimes Z.
\end{aligned}$$

Since X, Y, Z are orthogonal and so are all the different two-forms, the norm is given by

$$\begin{aligned}
\frac{1}{2} \|F_\omega\|^2 &= \dot{a}^2 g^{tt} g^{\rho\rho} + a^2 w^2 g^{tt} (g^{\theta\theta} + \sin^2 \theta g^{\varphi\varphi}) \\
&\quad + \dot{w}^2 g^{\rho\rho} (g^{\theta\theta} + g^{\varphi\varphi} \sin^2 \theta) + (1-w^2)^2 g^{\theta\theta} g^{\varphi\varphi} \sin^2 \theta \\
&= \frac{1}{r^2} \left[-e^{-2\tau} (\dot{a}^2 + 2a^2 w^2) + \frac{1}{r^2} (2\dot{w}^2 + (1-w^2)^2) \right]. \quad (4.7)
\end{aligned}$$

Next, to compute $\langle F_\omega \otimes F_\omega \rangle$ we work componentwise, so we note that

$$\begin{aligned}
F_{t\rho} &= -F_{\rho t} = -\dot{a}Z, & F_{t\theta} &= -F_{\theta t} = awY, & F_{t\varphi} &= -F_{\varphi t} = -aw \sin \theta X, \\
F_{\rho\theta} &= -F_{\theta\rho} = \dot{w}X, & F_{\rho\varphi} &= -F_{\varphi\rho} = \dot{w} \sin \theta Y, & F_{\theta\varphi} &= -F_{\varphi\theta} = -(1-w^2) \sin \theta Z.
\end{aligned}$$

Then, since the metric is diagonal, we have

$$\begin{aligned}
\langle F_\omega \otimes F_\omega \rangle &= \frac{1}{2} g^{\alpha\beta} \langle F_{\mu\alpha}, F_{\nu\beta} \rangle dx^\mu \otimes dx^\nu \\
&= \frac{1}{2} \left(g^{tt} \langle F_{\mu t}, F_{\nu t} \rangle + g^{\rho\rho} \langle F_{\mu\rho}, F_{\nu\rho} \rangle + g^{\theta\theta} \langle F_{\mu\theta}, F_{\nu\theta} \rangle + g^{\varphi\varphi} \langle F_{\mu\varphi}, F_{\nu\varphi} \rangle \right) dx^\mu \otimes dx^\nu.
\end{aligned}$$

Note that all terms are zero if μ and ν are different, due to the orthogonality of X, Y, Z . We have

$$\begin{aligned}\langle F_\omega \otimes F_\omega \rangle_{tt} &= \frac{\|F_{t\rho}\|^2}{2r^2} + \frac{\|F_{t\theta}\|^2}{2r^2} + \frac{\|F_{t\varphi}\|^2}{2r^2 \sin^2 \theta} = \frac{\dot{a}^2}{r^2} + \frac{2a^2 w^2}{r^2}, \\ \langle F_\omega \otimes F_\omega \rangle_{\rho\rho} &= -\frac{\|F_{\rho t}\|^2}{2e^{2\tau}} + \frac{\|F_{\rho\theta}\|^2}{2r^2} + \frac{\|F_{\rho\varphi}\|^2}{2r^2 \sin^2 \theta} = -e^{-2\tau} \dot{a}^2 + \frac{2\dot{w}^2}{r^2}, \\ \langle F_\omega \otimes F_\omega \rangle_{\theta\theta} &= -\frac{\|F_{\theta t}\|^2}{2e^{2\tau}} + \frac{\|F_{\theta\rho}\|^2}{2e^{2\tau}} + \frac{\|F_{\theta\varphi}\|^2}{r^2 \sin^2 \theta} = -e^{-2\tau} a^2 w^2 + \frac{\dot{w}^2}{r^2} + \frac{(1-w^2)^2}{r^2}.\end{aligned}$$

We conclude that

$$\begin{aligned}\langle F_\omega \otimes F_\omega \rangle &= \frac{1}{r^2} (\dot{a}^2 + 2a^2 w^2) dt \otimes dt + \left(\frac{2\dot{w}^2}{r^2} - e^{-2\tau} \dot{a}^2 \right) d\rho \otimes d\rho \\ &\quad + \left(\frac{\dot{w}^2}{r^2} + \frac{(1-w^2)^2}{r^2} - e^{-2\tau} a^2 w^2 \right) g_{\mathbb{S}^2}.\end{aligned}\tag{4.8}$$

4.2.3 Reduced equations

We can now finally write down the haunted EYM system (4.2a–4.2c). Inserting (4.3, 4.7, 4.8) into (4.2a) we thus see that the Einstein field equations are given by

$$\left\{ \begin{aligned} \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{r}\dot{\tau}}{r} - e^{-2\tau} (\dot{a}^2 + 2a^2 w^2) - \frac{1}{r^2} (2\dot{w}^2 + (1-w^2)^2) &= 0, \\ \ddot{r} + \dot{r}^2 - \frac{\dot{r}\dot{\tau}}{r} - \frac{2\dot{r}^2}{r^2} + \frac{2\ddot{r}}{r} - e^{-2\tau} (\dot{a}^2 - 2a^2 w^2) + \frac{1}{r^2} (2\dot{w}^2 - (1-w^2)^2) &= \dot{\phi}^2, \\ \frac{\ddot{r}}{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 + e^{-2\tau} \dot{a}^2 + \frac{(1-w^2)^2}{r^2} &= 0, \end{aligned} \right.$$

for the tt , $\rho\rho$ and $\mathbb{S}^2$ components, respectively. We may use the first and the third equation to simplify the second one, and replace it by

$$\frac{\dot{\phi}^2}{2} = 1 - e^{-2\tau} (\dot{a}^2 - 2a^2 w^2) + \frac{1}{r^2} (2\dot{w}^2 - (1-w^2)^2) - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right).\tag{4.10}$$

Next, we rewrite the Yang-Mills equations (4.6a, 4.6b) as

$$\left\{ \begin{aligned} \ddot{w} + \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{w} + w (r^2 e^{-2\tau} a^2 + (1-w^2)) &= 0, \\ \ddot{a} - \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{a} - 2aw^2 &= 0. \end{aligned} \right.$$

Finally, using (4.4, 4.7), we see that the phantom field equation (4.2c) reads

$$\ddot{\phi} + \left(\tau + \frac{\dot{r}}{r} \right) \dot{\phi} = 0,$$

which can be solved explicitly in terms of the metric coefficients to get

$$\phi(\rho) = \phi_0 + \sqrt{2} \pi_0 \int_0^\rho \frac{1}{r e^\tau}, \quad (4.12)$$

for some constants $\phi_0, \pi_0 \in \mathbb{R}$.³ We can insert this directly back into (4.10) to obtain

$$\frac{\pi_0^2}{(r e^\tau)^2} = 1 - e^{-2\tau} (\dot{a}^2 - 2a^2 w^2) + \frac{1}{r^2} (2\dot{w}^2 - (1 - w^2)^2) - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right),$$

thereby eliminating ϕ from the system. We have thus showed the following.

Theorem 4.2.1. *Under the ansätze*

$$\begin{aligned} g &= -e^{2\tau(\rho)} dt \otimes dt + r(\rho)^2 (d\rho \otimes d\rho + g_{\mathbb{S}^2}), \\ \omega &= w(\rho)(d\theta \otimes X + \sin \theta d\varphi \otimes Y) + (a(\rho) dt + \cos \theta d\varphi) \otimes Z, \\ \phi &= \phi(\rho), \end{aligned}$$

for $\tau, r, a, w \in C^2(\mathbb{R})$, the haunted Einstein-Yang-Mills equations (4.2a–4.2c) are equivalent to the system

$$\left\{ \begin{aligned} \ddot{w} + \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{w} + w(1 - w^2) + r^2 e^{-2\tau} w a^2 &= 0, & (4.13a) \\ \ddot{a} - \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{a} - 2a w^2 &= 0, & (4.13b) \\ \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{r}\dot{\tau}}{r} - e^{-2\tau} (\dot{a}^2 + 2a^2 w^2) - \frac{1}{r^2} (2\dot{w}^2 + (1 - w^2)^2) &= 0, & (4.13c) \\ \frac{\ddot{r}}{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 + e^{-2\tau} \dot{a}^2 + \frac{(1 - w^2)^2}{r^2} &= 0, & (4.13d) \\ 1 - e^{-2\tau} (\dot{a}^2 - 2a^2 w^2) + \frac{1}{r^2} (2\dot{w}^2 - (1 - w^2)^2) - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right) &= \frac{\pi_0^2}{(r e^\tau)^2}, & (4.13e) \end{aligned} \right.$$

where dot denotes differentiation with respect to ρ , and $\pi_0 \in \mathbb{R}$ is an arbitrary constant.

Remark. Note that ϕ does not appear explicitly in the equations (4.13a–4.13e) due to the general solution formula (4.12).

Prima facie, the system (4.13a–4.13e) is overdetermined since it consists of five equations for four unknowns. However, if one assumes that the first four equations (4.13a–4.13d) hold, then the final equation (4.13e) will automatically be satisfied provided that it holds at least at one point. We therefore view (4.13e) as a constraint equation for the initial conditions. This is somehow a more general feature of the spherically symmetric Einstein field equations [Kun91, §2], but in the present context one can most easily verify this simply by differentiating (4.13e).

³The constant ϕ_0 plays essentially no role and will be set to 0 in the future for simplicity, but the value of π_0 will be somewhat relevant later. The factor $\sqrt{2}$ appears merely to simplify the appearance of the future equations.

4.3 Wormhole initial conditions

Let us discuss appropriate initial conditions to supplement the equation system (4.13a–4.13e) with. Following §2.3, a wormhole spacetime has two asymptotically ends at $\rho \rightarrow \pm\infty$, which in particular requires the existence of at least one point at which r is stationary. Since r must be positive throughout the spacetime, we thus require

$$0 < r(0) = r_0 \in \mathbb{R} \quad \text{and} \quad \dot{r}(0) = 0.$$

On the other hand, the temporal coefficient τ is not constrained in any way other than the requirement that it should be finite throughout the spacetime (in particularly also at the asymptotically flat ends). We thus set

$$\tau(0) = \tau_0 \in \mathbb{R}, \quad \text{and} \quad \dot{\tau}(0) = \kappa_0 \in \mathbb{R}.$$

For all wormhole spacetimes considered in this manuscript, we will sacrifice some generality by making the additional assumption $\kappa_0 = 0$, because it makes life easier in a lot of cases, and we will mainly be concerned with studying symmetric wormholes, for which r and τ are even functions of ρ and the condition $\kappa_0 = 0$ is necessary.

The constraint equation (4.13e) evaluated at $\rho = 0$ reads

$$1 - e^{-2\tau_0} (b_0^2 - 2a_0^2 w_0^2) + \frac{1}{r_0^2} (2u_0^2 - (1 - w_0^2)^2) = \frac{\pi_0^2}{(r_0 e^{\tau_0})^2}, \quad (4.14)$$

where

$$(a, \dot{a})(0) = (a_0, b_0) \in \mathbb{R}^2 \quad \text{and} \quad (w, \dot{w})(0) = (w_0, u_0) \in \mathbb{R}^2$$

must satisfy

$$1 - e^{-2\tau_0} (b_0^2 - 2a_0^2 w_0^2) + \frac{1}{r_0^2} (2u_0^2 - (1 - w_0^2)^2) \geq 0$$

This shows the necessity of having $\pi_0 \neq 0$ in a wormhole spacetime, as otherwise this constraint would be only satisfied only for a few very constrained choices of initial values of the Yang-Mills potentials, none of which describe wormholes (this latter fact is not meant to be obvious at this point, but we will prove it later). In fact, since π_0 is arbitrary, we may keep τ_0 free and fix π_0 so as to satisfy (4.14).

We note finally that (4.13d) implies

$$\ddot{r}(0) = r_0 \left(1 - e^{-2\tau_0} b_0^2 - \frac{(1 - w_0^2)^2}{r_0^2} \right),$$

so that the initial point describes

- ◇ a wormhole throat if $e^{-2\tau_0} b_0^2 + \frac{(1 - w_0^2)^2}{r_0^2} < 1$, and
- ◇ a wormhole belly if $e^{-2\tau_0} b_0^2 + \frac{(1 - w_0^2)^2}{r_0^2} > 1$.

The case of equality requires a closer analysis and will be postponed until later chapters.

4.4 Classical solutions

Let us derive some classical wormhole solutions within the framework of Einstein-Yang-Mills theory. For the reader's convenience, we also summarize these solutions, together with the classical static spherically symmetric black hole solutions, in Table 4.1 (at the end of the chapter).

Assume that $w \equiv \pm 1$ and $a \equiv 0$. The equations (4.13a,4.13b) are then trivially satisfied and the connection ω is flat,⁴ i.e. $F_\omega \equiv 0$. The remaining equations of the system (4.13a–4.13e) reduce to

$$\left\{ \begin{array}{l} \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{\tau}\dot{\tau}}{r} = 0, \\ \frac{\ddot{r}}{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 = 0, \\ 1 - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right) = \frac{\pi_0^2}{(re^\tau)^2}. \end{array} \right. \quad \begin{array}{l} (4.15a) \\ (4.15b) \\ (4.15c) \end{array}$$

If we define $\kappa = \dot{\tau} + \frac{\dot{r}}{r}$, we get by summing (4.15a,4.15b) that

$$\dot{\kappa} + \kappa^2 - 1 = 0,$$

which is a separable equation and can be solved to obtain $\kappa(\rho) = \tanh(\rho)$ (recall that we are making the simplifying assumption $\dot{\tau}(0) = 0$, cf. §4.3). If we again define $N = \frac{\dot{r}}{r}$, we see from (4.15b) that

$$\dot{N} + \kappa N - 1 = 0,$$

which is solved by $N(\rho) = \kappa(\rho) = \tanh(\rho)$. This means that

$$\tau \equiv \tau_0, \quad r(\rho) = r_0 \cosh(\rho).$$

We set $\tau_0 = 0$, so that $e^\tau \equiv 1$. The constraint (4.15c) then requires $\pi_0 = r_0$. Thus

$$\phi(\rho) = \phi_0 + \sqrt{2} \arctan(\sinh \rho) = \phi_0 + \sqrt{2} \operatorname{gd}(\rho),$$

where we recall that gd is the so-called *Gudermann function*. This solution thus represents the Ellis-Bronnikov wormhole with metric

$$g = -dt \otimes dt + r_0^2 \cosh^2(\rho) (d\rho \otimes d\rho + g_{\mathbb{S}^2}),$$

cf. §1.1.

Now assume that $w \equiv 0$. Then (4.13b) reduces to

$$\ddot{a} - \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{a} = 0,$$

⁴In fact, the connection is then induced by the Maurer-Cartan connection on $\mathbf{SU}(2)$ pulled back to $\mathbf{SU}(2)/\mathbf{U}(1)$, cf. equation (3.10).

implying that

$$\dot{a} = \frac{Qe^\tau}{r}$$

for some constant $Q \in \mathbb{R}$ whose absolute value can be interpreted as the electric charge, cf. (3.13). The remaining equations of the system (4.13a–4.13e) then reduce to

$$\left\{ \begin{array}{l} \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{\tau}\dot{r}}{r} - \frac{1+Q^2}{r^2} = 0, \end{array} \right. \quad (4.16a)$$

$$\left\{ \begin{array}{l} \frac{\ddot{r}}{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 + \frac{1+Q^2}{r^2} = 0, \end{array} \right. \quad (4.16b)$$

$$\left\{ \begin{array}{l} 1 - \frac{1+Q^2}{r^2} - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right) = \frac{\pi_0^2}{(re^\tau)^2}. \end{array} \right. \quad (4.16c)$$

As above, we may define $\kappa = \dot{\tau} + \frac{\dot{r}}{r}$ which leads to $\kappa(\rho) = \tanh(\rho)$ (recall again that we are making the simplifying assumption $\dot{\tau}(0) = 0$, cf. §4.3) and thus also $re^\tau = r_0 e^{\tau_0} \cosh(\rho)$. Defining again $N = \frac{\dot{r}}{r}$, the remaining equations imply that

$$\dot{N} = 1 - \frac{1+Q^2}{r^2} - \tanh(\rho)N = \frac{\pi_0^2}{r_0^2 e^{2\tau_0}} \operatorname{sech}^2(\rho) + \tanh(\rho)N - N^2.$$

At $\rho = 0$ this reads

$$\pi_0^2 = e^{2\tau_0}(r_0^2 - Q^2 - 1),$$

so that these solutions only make sense when $Q^2 \leq r_0^2 - 1$. To solve this, put

$$\sigma = \arctan(\sinh \rho) = \operatorname{gd}(\rho).$$

Denoting by prime the derivative with respect to σ , the equation for r becomes

$$r'' = 2 \tan(\sigma) r' + (1 - \lambda^2) r, \quad \lambda = \frac{\sqrt{1+Q^2}}{r_0}$$

Aiming to eliminate the first-order term, we look for solutions of the form

$$r(\sigma) = r_0 \sec(\sigma) v(\sigma).$$

The function v then satisfies the initial value problem

$$v'' + \lambda^2 v = 0, \quad v(0) = 1, \quad v'(0) = 0,$$

which can easily be solved using elementary methods. We therefore get

$$\begin{aligned} r &= r_0 \sec(\sigma) \cos(\lambda \sigma) \\ &= r_0 \cosh(\rho) \cos(\lambda \operatorname{gd}(\rho)), \end{aligned}$$

and also

$$\tau = \log \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\cos(\lambda \operatorname{gd}(\rho))}.$$

Note that, in the extremal case $r_0^2 = 1 + Q^2$ (i.e. $\lambda = 1$), we have $r \equiv 1$. On the other hand, if $r_0^2 > 1 + Q^2$ (i.e. $0 < \lambda < 1$), we get

$$\frac{\dot{r}}{r} = \tanh(\rho) - \lambda \operatorname{sech}(\rho) \tan(\lambda \operatorname{gd}(\rho)) \rightarrow 1$$

as $\rho \rightarrow \infty$. The solutions with $r_0^2 > 1 + Q^2$ are therefore asymptotically flat and represent the wormhole analogue of the classical Reissner-Nordström family of black holes. Such solutions have already been obtained in [GGS09], albeit in a different coordinate system and in somewhat greater generality, in particular without the simplifying assumption $\dot{\tau}(0) = 0$ (cf. §4.3). The general family of wormhole solutions with $w \equiv 0$ does not seem to have a universally known name, so we will simply refer to it as the *abelian wormhole family*, since the gauge group of the Yang-Mills theory in this case degenerates to the abelian group $\mathbf{U}(1) \cong \mathbb{S}^1$, since ω is $\mathfrak{u}(1)$ -valued.

4.5 Uniqueness theorem

Our next goal is to show that the wormhole solutions of the system (4.13a–4.13e) with $a \not\equiv 0$ are in some sense trivial. This will allow us to assume that $a \equiv 0$ without loss of generality, simplifying the equation system considerably.

Theorem 4.5.1. *Assume that (w, a, τ, r) is a wormhole (cf. Definition 2.3.2) solution of the system (4.13a–4.13e) such that $a \not\equiv 0$. Then $w \equiv 0$.*

Remark. The result implies that the solution belongs to the classical abelian wormhole family, which we have presented in §4.4 and Table 4.1 under the additional assumption $\dot{\tau}(0) = 0$ for simplicity, although the family can be obtained explicitly even for $\dot{\tau}(0) \neq 0$, cf. [GGS09]. In view of this, we will for the rest of the work assume that $a \equiv 0$. This can be viewed as a theorem of “no-hair” type, since it essentially says the wormhole solutions with non-trivial electric potentials are completely determined by their electric Yang-Mills charge at infinity.

Proof. The proof technique is due to [BP92, §3], where the theorem is proven in the context of particle-like and black hole solutions. The argument is essentially the same but our proof requires some minor modifications owing to the introduction of the phantom field and because we do not necessarily have $a(0) = 0$, cf. [BP92, Lemma 2].

Since a wormhole solution is asymptotically flat with two ends and the system (4.13a–4.13e) is autonomous, we can assume without loss of generality that $\dot{\tau}(0) = 0$. We recall that asymptotic flatness requires that the temporal metric coefficient τ and the mass (2.5) both have finite limits as $\rho \rightarrow \pm\infty$, cf. Definition 2.3.1. Observe that the mass satisfies

$$\begin{aligned} \dot{m} &= \frac{\dot{r}}{2} \left(1 + \frac{\dot{r}^2}{r^2} - \frac{2\ddot{r}}{r} \right) \\ &= \frac{\dot{r}}{2r^2} \left[r^2 e^{-2\tau} (\dot{a}^2 + 2a^2 w^2) + 2\dot{w}^2 + (1 - w^2)^2 - \pi_0^2 e^{-2\tau} \right], \end{aligned}$$

where the last line follows by using (4.13e) to eliminate the $\dot{\tau}$ -term from (4.13d). If we integrate this identity, we get

$$\int_0^\infty \frac{\dot{r}}{2r^2} [r^2 e^{-2\tau} (\dot{a}^2 + 2a^2 w^2) + 2\dot{w}^2 + (1 - w^2)^2] = m_\infty - \frac{r_0}{2} + \frac{\pi_0^2}{2} \int_0^\infty \frac{\dot{r}}{r^2 e^{2\tau}}. \quad (4.17)$$

The last integral converges because $\tau \rightarrow \tau_\infty < \infty$ by asymptotic flatness, so that if $\bar{\rho}$ is chosen large enough that $\dot{r}$ is positive for $\rho \geq \bar{\rho}$, we get

$$\int_{\bar{\rho}}^\infty \frac{\dot{r}}{r^2 e^{2\tau}} \leq c \int_{\bar{\rho}}^\infty \frac{dr}{r^2} = \frac{c}{r(\bar{\rho})}.$$

Thus the right-hand side of (4.17) is finite since m_∞ is finite by assumption. Since all terms on the left-hand side of (4.17) are non-negative, their separate integrals must therefore all be finite. In particular, since $\frac{\dot{r}}{r} \rightarrow 1$ by asymptotic flatness, we get

$$\int_0^\infty r \dot{a}^2 < \infty, \quad \int_0^\infty r a^2 w^2 < \infty, \quad \int_0^\infty \frac{\dot{w}^2}{r} < \infty. \quad (4.18)$$

Now assume $a \not\equiv 0$. We will show that $w \equiv 0$, which will give the result. Aiming to reach a contradiction, we assume that $w \not\equiv 0$.

Note that the system (4.13a–4.13e) is invariant under $\rho \mapsto -\rho$ and $a \mapsto -a$. We can thus assume without loss of generality that $a(0) \geq 0$ and $\dot{a}(0) \geq 0$ by reflecting as necessary. Equation (4.13b) gives

$$(re^{-\tau} \dot{a})(\rho) = (re^{-\tau} \dot{a})(0) + 2 \int_0^\rho re^{-\tau} a w^2,$$

implying that a is monotone for $\rho \geq 0$ (note that if $a(0) = \dot{a}(0) = 0$, then trivially $a \equiv 0$, whereas in the other cases $a > 0$ for sufficiently small $\rho > 0$ and $\dot{a}$ must stay positive by the formula above). On the other hand, Cauchy-Schwarz gives

$$|a(\rho) - a(0)|^2 = \left| \int_0^\rho \dot{a} \right|^2 \leq \int_0^\rho r \dot{a}^2 \int_0^\rho \frac{1}{r}$$

The final integral is finite since $\frac{\dot{r}}{r} \rightarrow 1$ by asymptotic flatness, and the first integral is finite by (4.18). This implies that a is bounded, and, being monotone, it must tend to finite positive limit a_∞ . Now using Cauchy-Schwarz again, we get

$$\frac{1}{2} |w(\rho)^2 - w(\bar{\rho})^2|^2 = \left| \int_{\bar{\rho}}^\rho w \dot{w} \right|^2 \leq \int_{\bar{\rho}}^\rho r w^2 \int_{\bar{\rho}}^\rho \frac{\dot{w}^2}{r}.$$

The right-hand side tends to zero for $\rho, \bar{\rho} \rightarrow \infty$ by (4.18) since a stays away from zero at infinity. A simple Cauchy criterion argument⁵ then implies that w tends to some limit as $\rho \rightarrow \infty$, and since $r w^2$ is integrable, this limit can only be 0.

⁵For any unbounded increasing sequence ρ_n , we have $|w(\rho_n)^2 - w(\rho_m)^2| \rightarrow 0$ as $n, m \rightarrow \infty$, so that the sequence $w(\rho_n)^2$ is Cauchy. Since $\mathbb{R}$ is complete, the sequence must converge, i.e. $w(\rho_n)^2 \rightarrow L \in \mathbb{R}$ as $n \rightarrow \infty$. Here, L is independent of the chosen sequence, since if two sequences yielded different limits, we could arrange them into a new sequence and contradict the fact that our sequence is Cauchy. Then if we take sequences a_n and b_n such that $w(a_n)^2 \rightarrow \limsup w^2$ and $w(b_n)^2 \rightarrow \liminf w^2$ respectively, we see that $\limsup w^2 = \liminf w^2$ and hence $\lim w = L$.

Now write (4.13a) as a first order system

$$\dot{w} = rve^{-\tau}, \quad \dot{v} = -rwe^{-\gamma},$$

where we define

$$\gamma = \tau - \log \left(\frac{e^{2\tau}(1-w^2)}{r^2} + a^2 \right).$$

Note that this is well-defined for $\rho \geq \bar{\rho}$ if $\bar{\rho}$ is sufficiently large, because $w \rightarrow 0$ at infinity so the quantity in the parentheses is positive. Moreover, $\gamma \rightarrow -2\log a_\infty$ at infinity. Therefore,

$$rv^2 = \frac{e^{2\tau}\dot{w}^2}{r} \quad \text{and} \quad \frac{\dot{v}^2}{r} = rw^2e^{-2\gamma}$$

both have finite integrals near infinity by (4.18), so by an analogous argument as above we see that $v \rightarrow 0$ at infinity.

Consider now (for $\rho \geq \bar{\rho}$)

$$H = e^{-\gamma}w^2 + e^{-\tau}v^2.$$

Then $H \rightarrow 0$ as $\rho \rightarrow \infty$. Furthermore, we see that

$$-\frac{\dot{H}}{H} = \frac{e^{-\gamma}\dot{\gamma}w^2 + e^{-\tau}\dot{\tau}v^2}{e^{-\gamma}w^2 + e^{-\tau}v^2} = \dot{\gamma}x + \dot{\tau}(1-x),$$

where

$$x = \frac{e^{-\gamma}w^2}{e^{-\gamma}w^2 + e^{-\tau}v^2}, \quad 0 \leq x \leq 1.$$

It follows that

$$-\frac{\dot{H}}{H} \leq \max(\dot{\gamma}, \dot{\tau}). \tag{4.19}$$

Now observe that

$$\begin{aligned} r(\dot{\gamma} - \dot{\tau}) &= -r \frac{d}{d\rho} \log \left(\frac{e^{2\tau}(1-w^2)}{r^2} + a^2 \right) \\ &= -2e^{\gamma-\tau} \left[\frac{e^{2\tau}(1-w^2)}{r} \left(\dot{\tau} - \frac{\dot{r}}{r} \right) - e^{\tau}wv + ra\dot{a} \right]. \end{aligned}$$

The first term in the parentheses tends to zero by (4.13e), and the second term trivially also tends to zero because $w, v \rightarrow 0$. If we multiply (4.13b) by a and integrate by parts, we obtain

$$(ra\dot{a})(\rho) = (ra\dot{a})(0) + \int_0^\rho re^{-\tau} (\dot{a}^2 + 2a^2w^2) d\rho.$$

But the integral on the right-hand side converges for $\rho \rightarrow \infty$ by (4.18), so $ra\dot{a}$ tends to some positive limit by the monotone convergence theorem (recall also that we have ensured that $(a\dot{a})(0) \geq 0$). It follows that $r(\dot{\gamma} - \dot{\tau})$ converges to some negative limit, and hence $\dot{\gamma} \leq \dot{\tau}$ for sufficiently large ρ . In particular we obtain from (4.19) that $H \geq H(\bar{\rho})e^{\tau(\bar{\rho})-\tau}$ for all sufficiently large $\bar{\rho}$ and $\rho \geq \bar{\rho}$, which contradicts the fact that $H \rightarrow 0$ unless $H \equiv 0$. Hence, we must have $w \equiv 0$, completing the proof. $\square$

Name	Metric g , connection ω , phantom field ϕ	Parameters
Schwarzschild	$g = -\tanh^2\left(\frac{\rho}{2}\right) dt \otimes dt + r_0^2 \cosh^4\left(\frac{\rho}{2}\right) (d\rho \otimes d\rho + g_{S^2})$ $\omega = d\theta \otimes X + \sin\theta d\varphi \otimes Y + \cos\theta d\varphi \otimes Z$ $\phi \equiv 0$	$r_0 > 0$
Reissner-Nordström	$g = -\frac{(1-\lambda^2)^2 \tanh^2\left(\frac{\rho}{2}\right)}{[1-\lambda^2 \tanh^2\left(\frac{\rho}{2}\right)]^2} dt \otimes dt + r_0^2 \cosh^4\left(\frac{\rho}{2}\right) [1-\lambda^2 \tanh^2\left(\frac{\rho}{2}\right)]^2 (d\rho \otimes d\rho + g_{S^2})$ $\omega = \frac{Q}{r_0} [(1-\lambda^2) \cosh^2\left(\frac{\rho}{2}\right) + \lambda^2]^{-1} dt \otimes Z + \cos\theta d\varphi \otimes Z$ $\phi \equiv 0$	$r_0 > 0, Q \in \mathbb{R}$ $\lambda = \frac{\sqrt{1+Q^2}}{r_0}$ $0 < \lambda < 1$
Ellis-Bronnikov	$g = -dt \otimes dt + r_0^2 \cosh^2(\rho) (d\rho \otimes d\rho + g_{S^2})$ $\omega = d\theta \otimes X + \sin\theta d\varphi \otimes Y + \cos\theta d\varphi \otimes Z$ $\phi = \sqrt{2} \operatorname{gd}(\rho)$	$r_0 > 0$
Abelian wormhole	$g = -\frac{\cos^2\left(\frac{\pi\lambda}{2}\right)}{\cos^2(\lambda \operatorname{gd}(\rho))} dt \otimes dt + r_0^2 \cosh^2(\rho) \cos^2(\lambda \operatorname{gd}(\rho)) (d\rho \otimes d\rho + g_{S^2}),$ $\omega = \frac{Q \cos\left(\frac{\pi\lambda}{2}\right)}{r_0 \lambda} \tan(\lambda \operatorname{gd}(\rho)) dt \otimes Z + \cos\theta d\varphi \otimes Z$ $\phi = \sqrt{2(1-\lambda^2)} \operatorname{gd}(\rho)$	$r_0 > 0, Q \in \mathbb{R}$ $\lambda = \frac{\sqrt{1+Q^2}}{r_0}$ $0 < \lambda < 1$

Table 4.1: Some classical solutions within the haunted Einstein-Yang-Mills framework. Here, $\operatorname{gd}(\rho) = \arctan(\sinh \rho)$ is the Gudermannian function. Note that all these families are disconnected in our setting. There are two reasons for this. First, the wormhole solutions are disconnected from the black hole solutions since the wormhole solutions require the parameter π_0 to be of fixed prescribed value, whereas the black hole solutions have $\pi_0 = 0$. Second, the charged solutions are disconnected from the charge-free ones because our choice of gauge for the connection fixes the magnetic charge $P^2 = 1$. In a more general setting, one would have $\lambda = \sqrt{P^2 + Q^2}/r_0$ as in the most general Reissner-Nordström solution. However, if we formally set $\lambda = 0$, then the metrics of the charged solutions do reduce to the charge-free ones.

Chapter 5

Existence of haunted Einstein-Yang-Mills wormholes

In this chapter, we prove the existence of infinite sequences of wormholes in the haunted $\mathbf{SU}(2)$ Einstein-Yang-Mills theory. We recall that the static spherically symmetric haunted $\mathbf{SU}(2)$ EYM equations are given by (cf. Theorem 4.2.1)

$$\left\{ \begin{array}{l} \ddot{w} + \left(\dot{\tau} - \frac{\dot{r}}{r} \right) \dot{w} + w(1 - w^2) = 0, \\ \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{r}\dot{\tau}}{r} - \frac{2\dot{w}^2}{r^2} - \frac{(1 - w^2)^2}{r^2} = 0, \\ \ddot{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 + \frac{(1 - w^2)^2}{r^2} = 0, \\ 1 + \frac{2\dot{w}^2}{r^2} - \frac{(1 - w^2)^2}{r^2} - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right) = \frac{\pi_0^2}{(re^\tau)^2}, \end{array} \right. \quad \begin{array}{l} (5.1a) \\ (5.1b) \\ (5.1c) \\ (5.1d) \end{array}$$

where $\pi_0 \in \mathbb{R}$ is a free parameter. Here, we have made the additional assumption that the electric potential a vanishes identically, as we do not lose generality in doing so in view of Theorem 4.5.1.

We recall from Definition 2.3.2 that the metric coefficients (τ, r) will describe a wormhole spacetime provided that they satisfy the asymptotic flatness conditions

$$\lim_{\rho \rightarrow \pm\infty} r = \infty, \quad \lim_{\rho \rightarrow \pm\infty} m = m_\infty^\pm, \quad \lim_{\rho \rightarrow \pm\infty} \tau = \tau_\infty^\pm, \quad (5.2)$$

where $m_\infty^\pm$ and $\tau_\infty^\pm$ are finite, and m denotes the mass function (2.5). Such a wormhole spacetime is said to be symmetric if τ and r are even.

We recall also the definitions of the Yang-Mills charges

$$Q^\pm = \lim_{\rho \rightarrow \pm\infty} re^{-\tau} |\dot{a}|, \quad P^\pm = \lim_{\rho \rightarrow \pm\infty} |1 - w^2|,$$

cf. (3.13). Since we are assuming $a \equiv 0$, the electric charges $Q^\pm$ are trivially zero. We will search for solutions such that the magnetic charges $P^\pm$ are also zero, i.e.

$$\lim_{\rho \rightarrow \pm\infty} |w| = 1. \quad (5.3)$$

Solutions (τ, r, w) of the system (5.1a–5.1d) satisfying (5.2, 5.3) will be referred to as *haunted hairy $\mathbf{SU}(2)$ wormholes*.¹ Our main goal is then to show the following.

Theorem 5.0.1 (Symmetric existence theorem). *For each $r_0 > 0$, there exists a sequence*

$$\{(\tau^{(n)}, r^{(n)}, w^{(n)})\}_{n \geq 0}$$

of symmetric haunted hairy $\mathbf{SU}(2)$ wormholes with $r^{(n)}(0) = r_0$. The Yang-Mills potential $w^{(n)}$ has n zeros and is symmetric, i.e. even or odd, with the same parity as n . Furthermore:

- ◇ *For $r_0 \geq 1$, the wormholes have only a single trapping horizon at $\rho = 0$, describing a non-degenerate throat.*
- ◇ *For $r_0 < 1$ and even (resp. odd) n , the wormholes have a non-degenerate throat (resp. belly) at $\rho = 0$.*

5.1 Initial boundary value problem

Following the notation in [BFM94, §6], we now rewrite (5.1a–5.1d) as a first order system by substituting²

$$N = \frac{\dot{r}}{r}, \quad U = \frac{\dot{w}}{r}, \quad \kappa = \dot{\tau} + N, \quad \zeta = \frac{\pi_0}{r e^\tau},$$

so that the system transforms to

$$\begin{cases} \dot{r} = rN, & (5.4a) \end{cases}$$

$$\begin{cases} \dot{N} = 1 - \frac{(1 - w^2)^2}{r^2} - \kappa N, & (5.4b) \end{cases}$$

$$\begin{cases} \dot{w} = rU, & (5.4c) \end{cases}$$

$$\begin{cases} \dot{U} = -(\kappa - N)U - \frac{w(1 - w^2)}{r}, & (5.4d) \end{cases}$$

$$\begin{cases} \dot{\kappa} = 1 + 2U^2 - \kappa^2, & (5.4e) \end{cases}$$

$$\begin{cases} \dot{\zeta} = -\kappa\zeta, & (5.4f) \end{cases}$$

¹Strictly speaking, only the solutions for which w is not identically equal to ± 1 should be called hairy, but we will allow ourselves to also call these trivial cases hairy for simplicity.

²The authors of the cited paper actually derive the Einstein-Yang-Mills equations (with no phantom field) with respect to the Schwarzschild coordinate r , and then make a change to the ρ -coordinates in order to desingularize the system. Of course, this results in the same equations we have presented here, but with $\zeta \equiv 0$. Other than the fact that it was more convenient for us to work with the ρ -coordinates from the start, our presentation also somewhat demystifies their definition of the dependent variable κ . Indeed, it is defined directly in terms of the derivatives of the metric coefficients, and appears naturally in the constraint (5.5) arising from the field equations.

together with the constraint (5.1d)

$$\zeta^2 = 1 + N^2 + 2U^2 - \frac{(1 - w^2)^2}{r^2} - 2\kappa N. \quad (5.5)$$

Using this constraint, we can also rewrite (5.4b) as

$$\dot{N} = (\kappa - N)N - 2U^2 + \zeta^2 \quad (5.6)$$

$$= \frac{1}{2} \left(1 - N^2 - 2U^2 - \frac{(1 - w^2)^2}{r^2} + \zeta^2 \right). \quad (5.7)$$

The system (5.4a–5.4f) is regular as long as $r > 0$. Hence, for any choice of initial conditions with $r(0) > 0$, there exists a unique local (real) analytic solution to the problem. In view of the analysis in §4.3, we choose the initial conditions

$$\begin{aligned} r(0) &= r_0, & w(0) &= w_0, & \kappa(0) &= 0, \\ N(0) &= 0, & U(0) &= U_0, & \zeta(0) &= \sqrt{E_0}, \end{aligned} \quad (5.8)$$

where

$$E_0 = 1 + 2U_0^2 - \frac{(1 - w_0^2)^2}{r_0^2},$$

and the parameters (r_0, w_0, U_0) belong to the *set of admissible initial data*

$$\mathcal{J}_0 = \left\{ (r_0, w_0, U_0) \in \mathbb{R}^3 \mid r_0 > 0, |w_0| \leq 1, E_0 \geq 0 \right\}, \quad (5.9)$$

Here, we recall that the assumption $\kappa(0) = 0$ is made for simplicity, but in fact κ could more generally take any value at 0. The assumption $|w_0| \leq 1$ is also made for simplicity, since solutions exiting $|w| \leq 1$ will turn out to be ill-behaved. We recall also that the initial condition for ζ comes from the constraint (5.5). Since we will eventually want to transform back to the original variables (τ, r, w) , we finally wish to recall here that the initial value of the temporal metric coefficient τ is kept free and instead we fix the phantom charge

$$\pi_0 = r_0 e^{\tau_0} \sqrt{E_0}, \quad (5.10)$$

in order to remain compatible with $\zeta = \pi_0 / (r e^\tau)$.

Since the system (5.4a–5.4f) is analytic for $r > 0$, the solutions depend analytically on the initial conditions. As our particular choice of initial conditions depends continuously³ on the initial data $(r_0, w_0, U_0) \in \mathcal{J}_0$, we see that the solutions of the initial value problem also depend continuously on (r_0, w_0, U_0) .

³We cannot get more than continuity with respect to the parameters (r_0, w_0, U_0) for this particular setup (at least not on the whole $\mathcal{J}_0$), due to the square root in the initial condition for ζ . This can be remedied by replacing ζ by the dependent variable $\eta = \zeta^2$, which does not change the regularity of the equations, and at the same time does not necessitate any non-analytic relations between the initial parameters. However, continuity will be entirely sufficient for our purposes, so we prefer to keep ζ as the variable as it results in somewhat cleaner statements later, and also emphasizes the non-negativity of ζ^2 , which is a consequence of the nature of the phantom term.

Remark 5.1.1. Note that the equations (5.4a–5.4f) possess the symmetries

$$(w, U) \mapsto -(w, U) \quad \text{and} \quad (\rho, N, U, \kappa) \mapsto -(\rho, N, U, \kappa).$$

Thus, the general solution to the initial value problem with initial data $(r_0, w_0, U_0) \in \mathcal{I}_0$ satisfies the identities

$$\begin{aligned} & (r, N, w, U, \kappa, \zeta)(\rho, r_0, w_0, U_0) \\ &= (r, N, -w, -U, \kappa, \zeta)(\rho, r_0, -w_0, -U_0) \\ &= (r, -N, w, -U, -\kappa, \zeta)(-\rho, r_0, w_0, -U_0) \\ &= (r, -N, -w, U, -\kappa, \zeta)(-\rho, r_0, -w_0, U_0). \end{aligned}$$

Solutions with either $w_0 = 0$ or $U_0 = 0$ are therefore symmetric, since in that case r, ζ are even functions, while N, κ are odd. Moreover:

- ◊ If $U_0 = 0$, then w is even, so these are often referred to as *even* solutions.
- ◊ If $w_0 = 0$, then w is odd, so these are often called *odd* solutions.

In particular, we see here that $w_0 = U_0 = 0$ implies $w \equiv U \equiv 0$.

With these symmetries in mind, we see that it suffices to study the solutions for $\rho \geq 0$. In particular, to prove Theorem 5.0.1, we wish to show the existence of initial data of the form $(r_0, w_0, 0)$ and $(r_0, 0, U_0)$, for which the solution is defined for all $\rho \geq 0$ and the dependent variables have the following limits as $\rho \rightarrow \infty$:

$$r \rightarrow \infty, \quad r(1 - N^2) \rightarrow \beta, \quad r\zeta \rightarrow \alpha, \quad |w| \rightarrow 1,$$

where

$$0 < \alpha = \pi_0 e^{-\tau_\infty} < \infty \quad \text{and} \quad 0 \leq \beta = 2m_\infty < \infty.$$

We will for the most of the chapter actually work with general initial data $(r_0, w_0, U_0) \in \mathcal{I}_0$, and then specialize to the cases $w_0 = 0$ and $U_0 = 0$ in §5.6.

Before proceeding with the proof, we would like to present the results of our numerical investigation, analogous to the one performed in [Hau+14], albeit in a different coordinate system. In Table 5.1 we display the initial data describing symmetric hairy wormhole solutions with n -zeros of w . The corresponding plots of the dependent variables are given in Figure 5.3 and Figure 5.4 (at the end of the chapter).

5.2 Outline of the proof

The blueprint for the proof of Theorem 5.0.1 is heavily inspired by the aforementioned paper [BFM94], but we also use some techniques given in [Smo+91; SW93; SWY93; BP92; Mai05]. In most of these papers, the authors studied the Einstein-Yang-Mills system without a phantom field, which amounts to setting $\zeta \equiv 0$ in our equations (5.4a–5.4f), (5.5). The observant reader may have noticed that the equation (5.4f) for the phantom

n	w_0	U_0
0	1	0
1	0	2.82926051095
2	0.754658133388	0
3	0	2.049370662163
4	0.65449167421	0
5	0	1.879026239527
6	0.630231347337	0
7	0	1.8491787362
8	0.625889847857	0
9	0	1.844262704371

Table 5.1: Initial data describing symmetric wormholes for $r_0 = 0.75$.

term ζ is actually decoupled⁴ from the rest of the system (5.4a–5.4e), which might lead one to believe that the analysis of these equations is the same as in the aforementioned papers. But this is misleading, since ζ still appears in the constraint (5.5), and one uses this constraint repeatedly throughout the analysis - e.g. we will often use the alternate forms (5.6, 5.7) of the equation for N . In particular, one of the main difficulties here is that the function r (and in particular also the mass m) loses a monotonicity property⁵ and could even oscillate, which makes the behaviour of the solutions somewhat analogous to those of the apparently simpler Yang-Mills-dilaton system [Mai05].

The proof is based on a shooting method, in the sense that we consider infima over suitably chosen subsets of the set of initial data $\mathcal{I}_0$, with the expectation that these infima correspond to initial data describing the desired global solutions. We have already shortly described the idea in §1.3, but let us again mention some details about the three main ingredients of the proof:

- ◊ *Classification theorem* (§5.3): It turns out that the initial value problem admits three possible types of solutions, classified according to the behaviour of N . In fact, N plays a more important role than the other dependent variables, since it will turn out the solution can only stop existing if $N \rightarrow -\infty$, and even in that case the remaining dependent variables stay bounded. Of particular importance will be the dichotomy between the regions $N + \zeta < 0$ and $N + \zeta \geq 0$, which also demonstrates that the phantom field ζ affects the solutions in a non-trivial way. Indeed, any orbit entering the former region will turn out to be singular, whereas the orbits staying in the latter region will be well-defined for all $\rho \geq 0$. The strip $|w| \leq 1$ will also play a major role, and any orbit exiting it will also turn out to be singular. These facts will allow us to work in the region $\{N + \zeta \geq 0, |w| \leq 1\}$, in which the

⁴We have also already seen this in §4.2.3.

⁵When $\zeta \equiv 0$, then (5.6) implies that $\dot{N}$ is non-positive whenever $N = 0$, and in particular the region $N < 0$ then turns out to be invariant, so that once r starts decreasing, it keeps decreasing (see [BFM94] for more details).

solutions are generally well-behaved. We will then study the asymptotic behaviour of the dependent variables, depending on the behaviour of N (in particular the number of its zeros and its sign near infinity), and show that only a handful of cases can occur. One of the biggest difficulties will be the fact that, *prima facie*, we do not know whether the dependent variables even have limits at infinity, so that various techniques will be applied to extract these limits (note also Remark 5.2.2 below). An important tool for this will be a simple result called *Barbălat's lemma*, which does not seem to be all too well-known, so we provide its statement.

Lemma 5.2.1 (Barbălat's lemma [FW16]). *Suppose that $f : [0, \infty) \rightarrow \mathbb{R}$ is a uniformly continuous function such that*

$$\lim_{y \rightarrow \infty} \int_0^y f(x) dx \quad \text{exists and is finite.}$$

Then $f(x) \rightarrow 0$ as $x \rightarrow \infty$.

We recall that a simple sufficient condition for the uniform continuity of f is that its derivative is bounded (provided that f is differentiable).

- ◇ *Neighbourhood theorem* (§5.4): For each solution type, we will study the solutions with nearby initial data. Such a result will provide us with a mechanism of controlling which solution types the aforementioned infima can describe. The proof is mainly based on the fact that the solutions depend continuously on the initial data, but certain cases require a closer analysis (again, note also Remark 5.2.2 below).
- ◇ *Energy theorem* (§5.5): We study solutions with small resp. large initial energy E_0 . This will provide us with an upper and lower bound for the shooting method. The small case $E_0 = 0$ will follow by a simple analysis of the initial conditions. The case when E_0 is large will, however, be much more involved. In fact, this case has not been studied in the aforementioned citations, so the proof given here involves completely new techniques developed specifically for the problem corresponding to wormhole initial conditions.

With these three results in hand, the result will follow by a simple shooting method argument, the details of which we leave for §5.6.

Remark 5.2.2. We would like to point out a potential gap in some of the the proofs given in [BFM94] for the analogues of the above mentioned theorems. Therein, the authors study the asymptotic behaviour of the dependent variables using some heavy machinery of dynamical systems, namely the theory of structurally stable vector fields [AA88, §1.3] (particularly in their analogues of the classification and the neighbourhood theorem). This is done by viewing the Yang-Mills equation

$$\ddot{w} + (\kappa - 2N)\dot{w} + w(1 - w^2) = 0$$

as a perturbation of the Yang-Mills equation in the flat limit ($\kappa \rightarrow 1$, $N \rightarrow 1$) and the cylindrical limit ($\kappa \rightarrow 1$, $N \rightarrow 0$) i.e.,

$$\ddot{w} \pm \dot{w} + w(1 - w^2) = 0.$$

The latter equations are the so-called *Duffing type equations* [Duf18], which can be studied using the elementary theory of planar autonomous ODE systems. While this certainly provides a good heuristic overview of how the solutions should behave, the author of the present manuscript does not understand how one can mathematically apply the theory of structural stability in this context. Aside from some technical difficulties such as the fact that the vector field corresponding to the equation is tangential to (at least some points of) the boundary of any compact set containing the equilibria (whereas the theory assumes transversality at the boundary), the entire theory of structural stability only applies to *autonomous* perturbations of autonomous planar dynamical systems. On the other hand, the idea here is to consider $(\kappa - 2N + 1)\dot{w}$ as a small (for large ρ) perturbation of the flat Yang-Mills equation, with κ and N being interpreted as fixed externally given functions. But such a perturbation is clearly non-autonomous, so that the theory of structural stability cannot be applied directly. The author was also unable to find other references containing results that could be applied in this context. In view of this, the proofs given in the present work take on the more raw analytical approach as hinted above, which arguably also simplifies the proof(s). We would like to point out that the methods used in this work can also be applied in the context of the above mentioned particle-like and black hole settings, if this is indeed deemed necessary.

5.3 Classification theorem

The goal of this section is to show the following result, which can be viewed as an analogue of [BFM94, Theorem 16].

Theorem 5.3.1 (Classification theorem). *Any solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$ belongs to one of the following classes:*

- (i) *Singular: There exists a finite point $\rho_\infty > 0$ such that*

$$r \rightarrow 0, \quad N \rightarrow -\infty \quad \text{as} \quad \rho \rightarrow \rho_\infty,$$

and the remaining dependent variables remain bounded as $\rho \rightarrow \rho_\infty$.

- (ii) *Asymptotically cylindrical: The solution is defined for all $\rho \geq 0$, stays in the region $|w| \leq 1$, and the dependent variables have the following limits at infinity:*

$$r \rightarrow 1, \quad N \rightarrow 0, \quad \zeta \rightarrow 0, \quad \kappa \rightarrow 1.$$

Furthermore, either

- ◇ $r \equiv 1$ and $w \equiv 0$, or
- ◇ $r_0 < 1$, $(w, U) \rightarrow (0, 0)$ as $\rho \rightarrow \infty$, and w has infinitely many zeros.

(iii) *Asymptotically flat:* The solution is defined for all $\rho \geq 0$, stays in the region $|w| \leq 1$, and the dependent variables have the following limits at infinity:

$$r \rightarrow \infty, \quad r(1 - N^2) \rightarrow \beta, \quad r\zeta \rightarrow \alpha, \quad \kappa \rightarrow 1,$$

where $0 < \alpha < \infty$ and $0 \leq \beta < \infty$. Furthermore, either

- ◇ $r_0 > 1$ and $w \equiv 0$, or
- ◇ $(w, rU) \rightarrow (\pm 1, 0)$ as $\rho \rightarrow \infty$.

Remark. Note that $r\zeta = \pi_0 e^{-\tau}$, so that τ has a finite limit at infinity in case (iii) provided that $\pi_0 \neq 0$. By the constraint (5.5), this limit can be calculated as

$$\tau_\infty = -\frac{1}{2} \log \left[\frac{1}{\pi_0^2} \lim_{\rho \rightarrow \infty} r^2 (1 - 2\kappa N + N^2) \right],$$

but does not seem to admit a closed form in terms of the initial conditions.

Remark 5.3.2. We have already obtained some trivial solutions to the general equations in §4.4. Here we provide a short review of the trivial solutions relevant for the classification, in order to keep the proof of Theorem 5.3.1 self-contained. Note that for $w_0 \in \{-1, 0, 1\}$ and $U_0 = 0$, we have that w is identically constant $w \equiv w_0$ and hence also $U \equiv 0$. In this case we can explicitly solve

$$\kappa(\rho) = \tanh(\rho), \quad \zeta(\rho) = \sqrt{E_0} \operatorname{sech}(\rho).$$

The remaining non-trivial equation is the Riccati type equation

$$\dot{N} = E_0 \operatorname{sech}^2(\rho) + \tanh(\rho)N - N^2 = 1 - \frac{(1 - w_0^2)^2}{r^2} - \tanh(\rho)N.$$

If $(w_0, U_0) = (\pm 1, 0)$, then $w \equiv \pm 1$ and we can also get the explicit solutions

$$r(\rho) = r_0 \cosh(\rho), \quad N(\rho) = \tanh(\rho),$$

for any $r_0 > 0$. This solution describes the Ellis-Bronnikov wormhole, cf. §1.1. For solutions with $w_0 = U_0 = 0$, we have $w \equiv 0$, and $E_0 \geq 0$ in this case implies $r_0 \geq 1$. These solutions are given by (cf. Table 4.1)

$$r(\rho) = r_0 \cosh(\rho) \cos \left(\frac{1}{r_0} \arctan(\sinh \rho) \right).$$

There are two separate cases:

- ◇ If $r_0 = 1$, then $r \equiv 1$ and thus $N \equiv 0$. This solution is asymptotically cylindrical.

- ◊ If $r_0 > 1$, then the solutions are asymptotically flat. They represent the abelian wormhole family, cf. Table 4.1.

We can now begin the proof of Theorem 5.3.1. To simplify the statements of certain results, we say that a region $U \subset \mathbb{R}^6$ in the phase space is *(forward) invariant* if it has the following property: if there is a point $\rho_0 \geq 0$ such that the solution enters U at $\rho = \rho_0$, then it stays in U for all $\rho \geq \rho_0$, i.e.

$$[\exists \rho_0 \geq 0 : (r, N, w, U, \kappa, \zeta)(\rho_0) \in U] \Rightarrow [\forall \rho \geq \rho_0, (r, N, w, U, \kappa, \zeta)(\rho) \in U].$$

For a trivial example, we see from (5.4f) that the region $\zeta > 0$ is invariant.

Throughout the rest of the manuscript, we will make extensive use of certain energy functions related to the equations. In view of this, they deserve a proper definition.

Definition 5.3.3. The *energy* of the system (5.4c–5.4f) is defined as the function

$$E = 1 + 2U^2 - \frac{(1 - w^2)^2}{r^2} = 2\kappa N - N^2 + \zeta^2, \quad (5.11)$$

where the second equality follows from the constraint (5.5). The *autonomous energy* is defined as

$$F = 2\dot{w}^2 - (1 - w^2)^2 = r^2(E - 1). \quad (5.12)$$

We first derive some basic inequalities.

Lemma 5.3.4. Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. For all $\rho \geq 0$ for which the solution is defined, we have

$$\kappa \geq \tanh(\rho) \geq N, \quad 0 \leq \zeta \leq \sqrt{E_0} \operatorname{sech}(\rho), \quad \kappa + N \leq 2 + \sqrt{E_0} \operatorname{sech}(\rho).$$

Remark. The first set of inequalities implies that the temporal metric coefficient τ is non-decreasing, since $\kappa - N = \dot{\tau}$. This can also be derived directly from (5.1b). Looking at the last two inequalities, one might hope that the stronger inequality $\kappa + N \leq 2 + \zeta$ holds, but this is in fact not true. Indeed, our numerical results show that this inequality is violated when the initial energy E_0 is large.

Proof. To prove the first inequality, let $\xi = \frac{1-\kappa}{1+\kappa}$ and calculate

$$\dot{\xi} = -2\xi - U^2(1 + \xi)^2 \leq -2\xi$$

Thus $e^{2\rho}\xi(\rho)$ does not increase, and $\xi(0) = 1$ therefore implies $\xi(\rho) \leq e^{-2\rho}$, which can be rearranged to get

$$\kappa \geq \frac{1 - e^{-2\rho}}{1 + e^{-2\rho}} = \tanh(\rho).$$

The inequalities for ζ then follows easily because (5.4f) implies

$$\zeta = \zeta(0) \exp\left(-\int_0^\rho \kappa\right) \leq \sqrt{E_0} \exp\left(-\int_0^\rho \tanh(\rho)\right) = \sqrt{E_0} \operatorname{sech}(\rho).$$

Next, if $\nu = N - \tanh(\rho)$, then

$$\begin{aligned}\dot{\nu} &= -\frac{(1-w^2)^2}{r^2} - \kappa N + \tanh^2(\rho) \\ &\leq -\kappa\nu - [\kappa - \tanh(\rho)] \tanh(\rho) \leq -\kappa\nu,\end{aligned}$$

so that ν decreases in the region $\nu > 0$, and $\nu(0) = 0$ thus implies $\nu \leq 0$. Finally, we set

$$\eta = \kappa + N - 2 - \sqrt{E_0} \operatorname{sech}(\rho)$$

and calculate

$$\dot{\eta} = 1 + \zeta^2 - \frac{1}{4}(\kappa + N)^2 - \frac{3}{4}(\kappa - N)^2 + \sqrt{E_0} \tanh(\rho) \operatorname{sech}(\rho),$$

cf. [BFM94, Lemma 10]. In the region $\eta \geq 0$, we have

$$\begin{aligned}\kappa + N &\geq 2 + \sqrt{E_0} \operatorname{sech}(\rho), \\ \kappa - N &\geq 2 - 2N + \sqrt{E_0} \operatorname{sech}(\rho) \geq \sqrt{E_0} \operatorname{sech}(\rho),\end{aligned}$$

where we use the fact that $N \leq \tanh(\rho) \leq 1$. Hence, we get

$$\dot{\eta} \leq -\sqrt{E_0} \operatorname{sech}(\rho)[1 - \tanh(\rho)] < 0,$$

so that η decreases in the region $\eta \geq 0$, which yields the desired inequality since $\eta(0) < 0$. $\square$

Lemma 5.3.5. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. If the solution is defined for all $\rho \geq 0$, then*

$$\liminf_{\rho \rightarrow \infty} \kappa \geq 1 \quad \text{and} \quad \zeta \rightarrow 0.$$

Furthermore, if $U \rightarrow 0$, then $\kappa \rightarrow 1$.

Proof. The first part of the statement follows trivially from the inequalities in Lemma 5.3.4. For the last claim, let $\varepsilon > 0$, define $a_\varepsilon = \sqrt{1 + \varepsilon}$ and $\xi_\varepsilon = \frac{a_\varepsilon - \kappa}{a_\varepsilon + \kappa}$. A simple calculation yields

$$\dot{\xi}_\varepsilon = -2a_\varepsilon \xi_\varepsilon - (U^2 - \varepsilon)(1 + \xi_\varepsilon)^2.$$

If $U \rightarrow 0$, then $U^2 \leq \varepsilon$ and hence $\dot{\xi}_\varepsilon \geq -2a_\varepsilon \xi_\varepsilon$ for large ρ , which implies that $\limsup \kappa \leq a_\varepsilon$ for all $\varepsilon > 0$ and letting $\varepsilon \rightarrow 0$ shows that $\limsup \kappa \leq 1$, giving $\kappa \rightarrow 1$. $\square$

Next, we show that all the dependent variables behave well as long as N is bounded.

Lemma 5.3.6. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. Suppose that the solution is defined (at least) for $0 < \rho < \bar{\rho}$, and that N remains bounded as $\rho \rightarrow \bar{\rho}$. Then all the dependent variables remain bounded as $\rho \rightarrow \bar{\rho}$ and the solution can be continued across $\bar{\rho}$.*

Proof. Since N is bounded, it follows that r is also bounded by (5.4a). From Lemma 5.3.4, we directly see that ζ is bounded, and also

$$0 \leq \kappa \leq 2 + \sqrt{E_0} - N,$$

so κ is bounded as well. So it remains only to study w and U . For this we will use the energies from Definition 5.3.3. Note that the energy $E = 2\kappa N - N^2 + \zeta^2$ is bounded and hence the autonomous energy F is bounded as well. In particular, we see that w is bounded if and only if U is bounded, so it suffices to show that w is bounded at $\bar{\rho}$.

Aiming to reach a contradiction, assume that w is unbounded at $\bar{\rho}$. We first note that, since w is unbounded, it must enter the region $|w| > 1$ for some $0 \leq \rho_0 < \bar{\rho}$. In view of the symmetry $(w, U) \mapsto -(w, U)$ (cf. Remark 5.1.1), we can without loss of generality assume that $w > 1$. In this region, w is monotone since

$$\ddot{w} = -(\kappa - 2N)\dot{w} - w(1 - w^2) > -(\kappa - 2N)\dot{w},$$

so $\ddot{w}|_{\dot{w}=0} > 0$, and hence $\dot{w} > 0$ for $w > 1$. Thus $|w| \rightarrow \infty$, and hence by the Cauchy-Schwarz inequality

$$\begin{aligned} \sqrt{\int_{\bar{\rho}-\delta}^{\rho} \dot{w}^2} &\geq \frac{1}{\sqrt{\rho - \bar{\rho} + \delta}} \int_{\rho-\delta}^{\bar{\rho}} |\dot{w}| \\ &\geq \frac{|w(\rho) - w(\bar{\rho} - \delta)|}{\sqrt{\rho - \bar{\rho} + \delta}} \rightarrow \infty, \end{aligned}$$

as $\rho \rightarrow \bar{\rho}$, which implies that $\dot{w}$ is not square integrable near $\bar{\rho}$. We will reach the desired contradiction by showing that $\dot{w}$ is, in fact, square integrable near $\bar{\rho}$.

Note that $\dot{N} \rightarrow -\infty$ by (5.4b) since κ, N, r are bounded and $|w| \rightarrow \infty$. It follows that N decreases for $\bar{\rho} - \delta \leq \rho \leq \bar{\rho}$ if $\delta > 0$ is sufficiently small, and in particular the limit

$$\lim_{\rho \rightarrow \bar{\rho}} N = \bar{N}$$

exists. We now consider two cases separately:

- (i) If $\bar{N} < \frac{1}{2} \tanh(\bar{\rho})$, then we can choose $\delta > 0$ so small that $2N - \tanh(\rho) \leq -c$ for $\bar{\rho} - \delta \leq \rho \leq \bar{\rho}$ and some positive constant c . By Lemma 5.3.4 (i), we then have

$$\kappa - 2N \geq \tanh(\rho) - 2N \geq c \quad \text{for } \bar{\rho} - \delta \leq \rho \leq \bar{\rho},$$

and so

$$\dot{F} = -4(\kappa - 2N)\dot{w}^2 \leq -4c\dot{w}^2,$$

which implies that $\dot{w}$ is square integrable over $\bar{\rho} - \delta \leq \rho \leq \bar{\rho}$ since F is bounded.

- (ii) If $\bar{N} \geq \frac{1}{2} \tanh(\bar{\rho}) > 0$, then we can choose $\delta > 0$ so small that $N \geq \frac{1}{4} \tanh(\bar{\rho}) =: c$ for $\bar{\rho} - \delta \leq \rho \leq \bar{\rho}$. If we define $b = r^2(1 - N^2) = 2rm$, then b is bounded since r and N are, and we have

$$\dot{b} = 2r^2N \left(\frac{(1 - w^2)^2}{r^2} + (\kappa - N)N \right) \geq 2\varepsilon(1 - w^2)^2 = 2c(2\dot{w}^2 - F) \geq 4c\dot{w}^2 - \tilde{c}$$

for some constant $\tilde{c} > 0$, since $\kappa - N \geq 0$ by Lemma 5.3.4 and F is bounded. This implies that $\dot{w}$ is square integrable over $\bar{\rho} - \delta \leq \rho \leq \bar{\rho}$.

Thus, $\dot{w}$ is square integrable near $\bar{\rho}$ in both cases, which gives the desired contradiction. $\square$

On the other hand, the following result characterizes singular orbits.

Lemma 5.3.7. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{S}_0$.*

- (i) *The region $\{|w| > 1, w\dot{w} > 0\}$ is invariant and any solution that enters it also enters the region $N + \zeta < 0$.*
- (ii) *The region $N + \zeta < 0$ is invariant and any solution that enters it is singular, cf. Theorem 5.3.1 (i).*

Remark. Note that, contrarily, a solution that enters the region $N + \zeta < 0$ does not necessarily also enter $\{|w| > 1, w\dot{w} > 0\}$.

Proof. Suppose that the orbit enters the region $\{|w| > 1, w\dot{w} > 0\}$. In view of the symmetry $(w, U) \mapsto -(w, U)$ (cf. Remark 5.1.1), we can assume without loss of generality that there is a point $\rho_0 \geq 0$ with $w(\rho_0) > 1$ and $\dot{w}(\rho_0) > 0$. Note that for $w > 1$,

$$\ddot{w} = -(\kappa - 2N)\dot{w} - w(1 - w^2) > -(\kappa - 2N)\dot{w},$$

so $\ddot{w}|_{\dot{w}=0} > 0$, which shows that w must keep increasing and hence the region $\{w > 1, \dot{w} > 0\}$ is invariant, so $(w, \dot{w})$ remains there for all $\rho \geq \rho_0$. Next, we want to show that the orbit enters $N + \zeta < 0$, so we study the orbit while it resides in the region $N + \zeta \geq 0$. This implies that $1 \geq \tanh(\rho) \geq N \geq -\zeta \geq -\sqrt{E_0} \operatorname{sech}(\rho)$ by Lemma 5.3.4. Note that the orbit exists as long as it stays in this region in view of Lemma 5.3.6. By Lemma 5.3.4, we have

$$\kappa \leq 2 + \sqrt{E_0} \operatorname{sech}(\rho) - N \leq 2 + 2\sqrt{E_0} \operatorname{sech}(\rho).$$

Put $T = (w^2 - 1)/r > 0$ and calculate (cf. [BFM94, Proposition 11])

$$\begin{aligned} \frac{d}{d\rho} \log |TU| &= w \frac{2U^2 + T^2}{TU} - \kappa \\ &\geq 2\sqrt{2} - \kappa \\ &\geq 2 \left[\sqrt{2} - 1 - \sqrt{E_0} \operatorname{sech}(\rho) \right], \end{aligned}$$

where we use the fact that

$$(\sqrt{2}U \pm T)^2 \geq 0 \quad \Rightarrow \quad 2\sqrt{2}|TU| \leq 2U^2 + T^2.$$

Thus, $|TU|$ increases strictly and uniformly for sufficiently large ρ , so the solution eventually reaches (and stays in) the region $|TU| \geq 1/\sqrt{2}$, implying also that $2U^2 + T^2 \geq 2$. In this region, (5.7) gives

$$\dot{N} = \frac{1}{2}(1 - N^2 - 2U^2 - T^2 + \zeta^2) \leq -\frac{1}{2}[1 - E_0 \operatorname{sech}^2(\rho)],$$

so N uniformly decreases for large ρ , and thus it eventually reaches the region $N + \zeta < 0$ (recall from Lemma 5.3.4 that ζ approaches zero), proving (i).

Now to prove (ii), put $\xi = N + \zeta$ and calculate

$$\dot{\xi} = -\xi^2 + (k + 2\zeta)\xi - 2\kappa\zeta - 2U^2.$$

Since $\kappa, \zeta \geq 0$, we see that ξ decreases in the region $\xi < 0$ and consequently this region is preserved once reached. In particular, if the orbit enters it, we have $\dot{\xi} \leq -\xi^2$, implying that $\xi \rightarrow -\infty$ at some finite point $\rho = \rho_\infty$, which in turn implies that $N \rightarrow -\infty$ as ζ is bounded (note that none of the other variables can explode before $N \rightarrow -\infty$ in view of Lemma 5.3.6).

Finally, we show that the other dependent variables remain bounded near the singular point ρ_∞ . This will also imply that $r \rightarrow 0$ at ρ_∞ e.g. by (5.4b). To this end, we adapt the techniques from [BFM94, Proposition 13].

Since w is trivially bounded if it remains in the strip $|w| \leq 1$ for all $\rho < \rho_\infty$, we consider only the case when w enters the invariant region $\{|w| > 1, w\dot{w} > 0\}$, and, as above, we assume without loss of generality that $w > 1, \dot{w} > 0$. Put $\eta = -r(N + \zeta) = -r\xi$. For sufficiently small δ , we have $N + \zeta < 0$ and thus $\eta > 0$ for $\rho_\infty - \delta < \rho < \rho_\infty$. We will show that $w\eta^{-\varepsilon}$ is bounded near ρ_∞ for $0 < \varepsilon < \frac{1}{2}$. This will imply that w is bounded because the constraint (5.5) gives

$$\begin{aligned} \eta^2 &= r^2(N^2 - \zeta^2) - 2r\zeta\eta \\ &\leq -r^2(1 + 2U^2 - 2\kappa N) + (1 - w^2)^2 \\ &\leq (1 - w^2)^2, \end{aligned}$$

so that $w\eta^{-\varepsilon} \geq w|1 - w^2|^{-\varepsilon}$. A simple calculation yields

$$\dot{\eta} = 2rU^2 + \eta\zeta + r\kappa(\zeta - N) > 0,$$

so that η increases and in particular stays away from zero near ρ_∞ . We have

$$\begin{aligned} w\eta^{-\varepsilon}(\rho) - w\eta^{-\varepsilon}(\rho_\infty - \delta) &= \int_{\rho_\infty - \delta}^{\rho} \frac{d}{d\rho}(w\eta^{-\varepsilon}) \\ &= \int_{\rho_\infty - \delta}^{\rho} \dot{w}\eta^{-\varepsilon} - \int_{\rho_\infty - \delta}^{\rho} w\eta^{-\varepsilon}\dot{\eta} \\ &\leq \int_{\rho_\infty - \delta}^{\rho} \dot{w}\eta^{-\varepsilon}, \end{aligned}$$

where the last inequality follows since $w > 1$ and $\dot{\eta} > 0$. Now by the Cauchy-Schwarz inequality, we get

$$\left(\int_{\rho_\infty - \delta}^{\rho} \dot{w}\eta^{-\varepsilon} \right)^2 \leq \int_{\rho_\infty - \delta}^{\rho} rU^2\eta^{-1-\varepsilon} \int_{\rho_\infty - \delta}^{\rho} r\eta^{1-\varepsilon},$$

so it suffices to show that the two integrals on the right-hand side are finite as $\rho \rightarrow \rho_\infty$. For the first integral, we can estimate (because η increases)

$$\begin{aligned} 2|\eta(\rho_\infty - \delta)|^{-\varepsilon} &\geq \left| \int_{\rho_\infty - \delta}^{\rho_\infty} \frac{d}{d\rho} \eta^{-\varepsilon} \right| \\ &= \varepsilon \int_{\rho_\infty - \delta}^{\rho_\infty} (2rU^2\eta^{-1-\varepsilon} + \zeta\eta^{-\varepsilon} + r\kappa(\zeta - N)\eta^{-1-\varepsilon}), \end{aligned}$$

and since all the integrands on the right hand side are non-negative, their separate integrals must all be finite, in particular the one involving U . For the second integral, we write

$$\eta = -\frac{r^{1-\varepsilon}}{\varepsilon} \frac{d}{d\rho} r^\varepsilon - r\zeta = -r^{1-\varepsilon} \left(\frac{1}{\varepsilon} \frac{d}{d\rho} r^\varepsilon + r^\varepsilon \zeta \right),$$

so that the integral of η near ρ_∞ is finite, and Hölder's inequality implies

$$\int_{\rho_\infty - \delta}^{\rho_\infty} r\eta^{1-\varepsilon} \leq \left(\int_{\rho_\infty - \delta}^{\rho_\infty} r^{\frac{1}{\varepsilon}} \right)^\varepsilon \left(\int_{\rho_\infty - \delta}^{\rho_\infty} \eta \right)^{1-\varepsilon},$$

and the latter is finite since $r > 0$ is decreasing. Thus, w is bounded near ρ_∞ , as desired.

Now we calculate

$$\frac{d}{d\rho}(r^\varepsilon U) = -(\kappa - (1 + \varepsilon)N)r^\varepsilon U - \frac{w(1 - w^2)}{r^{1-\varepsilon}}.$$

Setting

$$\xi = \int_{\rho_\infty - \delta}^{\rho} (\kappa - (1 + \varepsilon)N),$$

we see that ξ increases and $\xi > 0$, so that

$$\begin{aligned} r^\varepsilon |U(\rho)| &\leq e^{-\xi(\rho)} r^\varepsilon |U(\rho_\infty - \delta)| + e^{-\xi(\rho)} \int_{\rho_\infty - \delta}^{\rho} \frac{e^\xi |w(1 - w^2)|}{r^{1-\varepsilon}} \\ &\leq r^\varepsilon |U(\rho_\infty - \delta)| + c \int_{\rho_\infty - \delta}^{\rho} \frac{1}{r^{1-\varepsilon}}. \end{aligned}$$

But since r decreases and η increases, we have (for $\varepsilon < 1$)

$$\begin{aligned} \int_{\rho_\infty - \delta}^{\rho} \frac{1}{r^{1-\varepsilon}} &= \int_{\rho_\infty - \delta}^{\rho} \frac{1}{\eta} \left(r^\varepsilon \zeta - \frac{1}{\varepsilon} \frac{d}{d\rho} r^\varepsilon \right) \\ &\leq \frac{1}{\eta(\rho_\infty - \delta)} \left(\int_{\rho_\infty - \delta}^{\rho} r^\varepsilon \zeta + \frac{r(\rho_\infty - \delta)^\varepsilon}{\varepsilon} - \frac{r(\rho)^\varepsilon}{\varepsilon} \right), \end{aligned}$$

and we see that this integral is finite as $\rho \rightarrow \rho_\infty$. It follows also that $r^\varepsilon |U|$ is bounded near ρ_∞ . Thus,

$$\int_{\rho_\infty - \delta}^{\rho_\infty} |U|^n = \int_{\rho_\infty - \delta}^{\rho_\infty} r^{-n\varepsilon} (r^\varepsilon |U|)^n \leq c \int_{\rho_\infty - \delta}^{\rho_\infty} r^{-n\varepsilon},$$

and the latter integral is finite if $\varepsilon < 1/n$, by the same argument as above. Hence, $|U|^n$ is integrable near ρ_∞ for any $n \geq 0$, and it follows from (5.4e) that κ is bounded near ρ_∞ . It only remains to show that U is bounded. If we define

$$V = NU - \frac{w(1-w^2)}{r},$$

we see that $\dot{U} = -\kappa U + V$, so the boundedness of U will follow if we can show that V is bounded. We calculate

$$\dot{V} = (3w^2 - 1 - 2U^2 + \zeta^2)U,$$

so that the boundedness of V follows from the boundedness of w and ζ , combined with the integrability of $|U|^n$. This completes the proof. $\square$

In view of Lemma 5.3.7, we may assume for the rest of the proof of Theorem 5.3.1 that the solution remains in the regions $N + \zeta \geq 0$ and $|w| \leq 1$.

Lemma 5.3.8. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. Assume the solution satisfies $N + \zeta \geq 0$ and $|w| \leq 1$ for all $\rho \geq 0$ for which it exists. Then the solution is well-defined for all $\rho \geq 0$, and furthermore:*

- (i) *the integral $\int_0^\infty N$ exists and has finite negative part,*
- (ii) *r has a non-zero limit at infinity, which is finite if and only if $N \rightarrow 0$,*
- (iii) *all other dependent variables remain bounded as $\rho \rightarrow \infty$.*

Remark. We would also like to point out that, a priori, N could oscillate, i.e. the integral could be of the form $\infty - \infty$, so the existence in (i) is a non-trivial matter. In part (iii), we make no claims about the existence of limits at infinity - this will be studied in the subsequent lemmata.

Proof. By the bounds in Lemma 5.3.4, we see that $|N|$ stays bounded as long as $N + \zeta \geq 0$, and therefore the solution is well-defined for all $\rho \geq 0$ by Lemma 5.3.6.

By the monotone convergence theorem⁶ and Lemma 5.3.4, we have

$$\lim_{\rho \rightarrow \infty} \int_0^\rho \zeta = \int_0^\infty \zeta \leq \sqrt{E_0} \int_0^\infty \operatorname{sech}(\rho) = \frac{\pi \sqrt{E_0}}{2} < \infty.$$

⁶Strictly speaking, we should first apply the theorem to conclude that for any increasing sequence $\rho_n \rightarrow \infty$, we have (denoting by χ_A the characteristic function of a set A)

$$\lim_{n \rightarrow \infty} \int_0^{\rho_n} \zeta = \lim_{n \rightarrow \infty} \int_0^\infty \chi_{(0, \rho_n)} \zeta = \int_0^\infty \zeta,$$

which also implies the existence of the limit with respect to the continuous variable ρ .

On the other hand, since $N + \zeta \geq 0$ by assumption, we also see by the monotone convergence theorem that

$$\lim_{\rho \rightarrow \infty} \int_0^\rho (N + \zeta) = \int_0^\infty (N + \zeta)$$

where the integral on the right-hand side could be infinite. Thus,

$$\lim_{\rho \rightarrow \infty} \int_0^\rho N = \lim_{\rho \rightarrow \infty} \left[\int_0^\rho (N + \zeta) - \int_0^\rho \zeta \right] = \int_0^\infty (N + \zeta) - \int_0^\infty \zeta = \int_0^\infty N,$$

where we may take the limit on each term separately since they both have definite sign, and the negative part, i.e. the integral of ζ , has finite limit. Note that this also implies that the negative part of the integral of N is finite and in particular the (Lebesgue) integral of N over $[0, \infty)$ exist (but could be infinite).

Now equation (5.4a) implies

$$r(\rho) = r_0 \exp \int_0^\rho N \rightarrow r_0 \exp \int_0^\infty N,$$

so that r has a limit at infinity, which is non-zero because the integral of N cannot be negatively infinite. For the second claim in (ii), note that if $N \rightarrow 0$, then the constraint (5.5) shows that

$$\begin{aligned} \liminf_{\rho \rightarrow \infty} \frac{(1 - w^2)^2}{r^2} &= \liminf_{\rho \rightarrow \infty} (1 - 2\kappa N + N^2 + 2U^2 - \zeta^2) \\ &\geq 1 + \liminf_{\rho \rightarrow \infty} 2U^2 + \lim_{\rho \rightarrow \infty} (-2\kappa N + N^2 - \zeta^2) \\ &= 1 + \liminf_{\rho \rightarrow \infty} 2U^2 \geq 1, \end{aligned}$$

since also $\zeta \rightarrow 0$ and κ is bounded due to the inequality $\kappa + N \leq 2 + \sqrt{E_0}$. This implies that r cannot be unbounded (note that $|w| \leq 1$ for $N + \zeta \geq 0$ by Lemma 5.3.6) and hence has a finite limit by the preceding part of the lemma. On the other hand, if r has a finite limit, then so does $\log \frac{r}{r_0} = \int_0^\rho N$, and since $\dot{N}$ is bounded by (5.4b), it follows that $N \rightarrow 0$ by Barbălat's lemma (Lemma 5.2.1).

For (iii), we first note that that $\kappa \leq 2 + \sqrt{E_0} - N$ is bounded since $-\zeta \leq N \leq 1$, where the first inequality follows by assumption and the second inequality follows from Lemma 5.3.4. It follows that the energy (5.11) is bounded. Furthermore, $|w| \leq 1$ because the assumption $N + \zeta \geq 0$ implies that w cannot exit this region, cf. Lemma 5.3.7 (i). Since r stays away from zero at infinity by the already proven part of the lemma, we see that $\frac{(1-w^2)^2}{r^2}$ is bounded, and thus so is $2U^2 = \frac{(1-w^2)^2}{r^2} - 1 + E$. $\square$

Now we wish to consider three cases separately:

- (i) N has infinitely many zeros,

- (ii) N has finitely many zeros and $N < 0$ after the last zero,
- (iii) N has finitely many zeros and $N > 0$ after the last zero.

Before dealing with each of these cases, we would like to prove some preparatory results. First, we provide a lemma which would thematically be a better fit for the next section, but we will use it in this section as well, so we state and prove it somewhat ahead of time.

Lemma 5.3.9. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{J}_0$. If $r_0 \geq 1$, then either*

- $\diamond r \equiv 1$ and $w \equiv 0$, or
- $\diamond N > 0$ for all $\rho > 0$ such that $|w| \leq 1$.

Proof. If $(r_0, w_0, U_0) = (1, 0, 0)$, then the solution is trivial with $r \equiv 1$ and $w \equiv 0$, cf. Remark 5.3.2. Assume therefore that $r_0 \geq 1$ and $(r_0, w_0, U_0) \neq (1, 0, 0)$. By (5.4b), we have

$$\dot{N}(0) = 1 - \frac{(1 - w_0^2)^2}{r_0^2} \geq 1 - (1 - w_0^2)^2 \geq 0.$$

If $r_0 > 1$, then the first inequality is strict and $N > 0$ for small $\rho > 0$ since $N(0) = 0$ by our choice of initial conditions. If $r_0 = 1$ and $|w_0| \neq 0$, then the second inequality is strict and again $N > 0$ for small $\rho > 0$. If $r_0 = 1$ and $w_0 = 0$, then we can continue differentiating (5.4b) to see that $\dot{N}(0) = \ddot{N}(0) = 0$ but $\ddot{\ddot{N}}(0) = 4U_0^2$, and the latter is positive since otherwise $(r_0, w_0, U_0) = (1, 0, 0)$. Thus, $N > 0$ for small $\rho > 0$ in all cases.

Suppose that N ever reaches zero again in the region $|w| \leq 1$, so that there exists $\bar{\rho} > 0$ with $N > 0$ and $|w| \leq 1$ for $0 < \rho < \bar{\rho}$, as well as $N(\bar{\rho}) = 0$. Then r increases on this range and $r(\bar{\rho}) > r_0 \geq 1$, giving, by the same estimate as above, that $\dot{N}(\bar{\rho}) > 0$. This is a contradiction, so that we must have $N > 0$ for all $\rho > 0$. $\square$

The next result tells us that the condition $N \rightarrow 0$ as $\rho \rightarrow \infty$ is in fact a characterizing property of asymptotically cylindrical orbits.

Lemma 5.3.10. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{J}_0$. If $N \rightarrow 0$ as $\rho \rightarrow \infty$, then the solution is asymptotically cylindrical, cf. Theorem 5.3.1 (ii).*

Proof. We first note that r has a finite limit $0 < r_\infty \leq 1$ by Lemma 5.3.8 (ii). Furthermore, the energy (5.11) tends to 0 (since κ is bounded and $\zeta \rightarrow 0$). Now the autonomous energy (5.12) satisfies $F = r^2(E - 1) \rightarrow -r_\infty^2$. For any $\rho_0 \geq 0$, we also have

$$F(\rho_0) + F(\rho) = - \int_{\rho_0}^{\rho} \dot{F} = 4 \int_{\rho_0}^{\rho} (\kappa - 2N) \dot{w}^2.$$

If ρ_0 is selected so large that $\kappa - 2N \geq c$ for $\rho \geq \rho_0$ and some constant $c > 0$ (this is possible since $\liminf \kappa \geq 1$ and $N \rightarrow 0$), then we see by the monotone convergence

theorem that

$$\int_{\rho_0}^{\infty} \dot{w}^2 = \lim_{\rho \rightarrow \infty} \int_{\rho_0}^{\rho} \dot{w}^2 \leq \frac{1}{4c} \lim_{\rho \rightarrow \infty} (F(\rho_0) + F(\rho)) = \frac{F(\rho_0) + r_{\infty}^2}{4c},$$

and hence $\dot{w}$ is square-integrable near infinity. Since

$$\ddot{w} = -(\kappa - 2N)\dot{w} - w(1 - w^2)$$

is bounded (note that $\dot{w}$ is bounded because F and w are), we see that $\dot{w}^2$ is uniformly continuous, and hence $\dot{w} \rightarrow 0$ by Barbălat's lemma (Lemma 5.2.1). Thus $U \rightarrow 0$, as well as $\kappa \rightarrow 1$ by Lemma 5.3.5.

Now $(1 - w^2)^2 = 2\dot{w}^2 - F$ has a limit at infinity, and hence w also tends to some limit $|w_{\infty}| \leq 1$ by continuity. By equation (5.4c), we must have $w_{\infty} \in \{0, \pm 1\}$ since $\ddot{w} \rightarrow -w_{\infty}(1 - w_{\infty}^2)$, and any other choice of w_{∞} would contradict $\dot{w} \rightarrow 0$. But $w_{\infty} = \pm 1$ is impossible since that would imply the absurdity $F \rightarrow 0 = -r_{\infty}^2$. It follows that $w \rightarrow 0$, and from $E \rightarrow 0$ we also get $r \rightarrow 1$.

Next, we observe that the only asymptotically cylindrical solution with $r_0 \geq 1$ is the trivial orbit with $r \equiv 1$ and $w \equiv 0$ by Lemma 5.3.9 (since otherwise $N > 0$ and $r > 1$ for $\rho > 0$ with $|w| \leq 1$, and we cannot have $r \rightarrow 1$). On the other hand, for $r_0 < 1$ the trivial solution $w \equiv 0$ has $E_0 < 0$ and thus is not admissible. Therefore it only remains to show that w truly has infinitely many zeros in the case $r_0 < 1$. To this end, we consider the polar angle defined by

$$-2\pi < \theta(0) \leq 0, \quad \tan \theta = \frac{\dot{w}}{w} \quad \text{if } w \neq 0,$$

and extended smoothly across zeros of w . Note that this is well-defined because $(w, \dot{w})$ stays away from the origin, since $w \not\equiv 0$. A simple calculation yields

$$\dot{\theta} + 1 = -(\kappa - 2N) \frac{w\dot{w}}{w^2 + \dot{w}^2} + \frac{w^4}{w^2 + \dot{w}^2}. \quad (5.13)$$

Now since $2|w\dot{w}| \leq w^2 + \dot{w}^2$,

$$\frac{|w\dot{w}|}{w^2 + \dot{w}^2} \leq \frac{1}{2} \quad \text{and} \quad \frac{w^4}{w^2 + \dot{w}^2} = \frac{w^2}{1 + (\dot{w}/w)^2} \leq w^2$$

we get

$$\dot{\theta} + \frac{1}{2} \leq \frac{1}{2}|\kappa - 2N - 1| + w^2.$$

But the right hand side of the latter inequality tends to 0, so $\limsup \dot{\theta} \leq -\frac{1}{2}$, implying that $\theta \rightarrow -\infty$, and thus w necessarily crosses zero infinitely many times. $\square$

With Lemma 5.3.10 in hand, we can now continue the proof of Theorem 5.3.1, by tackling the different cases depending on the number of zeros of N , as already hinted.

Lemma 5.3.11. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. If N has infinitely many zeros, then the solution is asymptotically cylindrical, cf. Theorem 5.3.1 (ii).*

Proof. Let ρ_n be the increasing sequence of zeros of N . Note that ρ_n must be unbounded by Lemma 5.3.6 and the fact that the region $N < -\zeta$ is invariant, so that N can only have zeros while staying in the region $-\zeta \leq N \leq 1$. Since r tends to some limit at infinity by Lemma 5.3.8 (ii), we may use the constraint (5.5) to calculate

$$\begin{aligned} \lim_{\rho \rightarrow \infty} r(\rho)^2 &= \lim_{n \rightarrow \infty} r(\rho_n)^2 \\ &= \lim_{n \rightarrow \infty} \frac{(1 - w(\rho_n)^2)^2}{1 + 2U(\rho_n)^2 - \zeta(\rho_n)^2} \\ &\leq \limsup_{n \rightarrow \infty} (1 - w(\rho_n)^2)^2 \leq 1 \end{aligned}$$

which is finite since w is bounded by Lemma 5.3.8 (iii). This implies that the limit of r is finite, and thus $N \rightarrow 0$ by Lemma 5.3.8 (ii). Hence, we may apply Lemma 5.3.10 to conclude. $\square$

Lemma 5.3.12. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. Assume there is a point $\rho_0 \geq 0$ such that $N(\rho_0) = 0$ and $N(\rho) < 0$ for $\rho > \rho_0$. Then the solution is singular, cf. Theorem 5.3.1 (i).*

Proof. In view of Lemma 5.3.7, we only need to show that N exits the region $-\zeta \leq N < 0$. Aiming to reach a contradiction, assume that N remains in this region for all $\rho > \rho_0$. Then we also have $|w| \leq 1$ in view of Lemma 5.3.7 (i). Furthermore, we see that $N \rightarrow 0$ as $\rho \rightarrow \infty$ (since $\zeta \rightarrow 0$), and hence $r \rightarrow 1$ by Lemma 5.3.10. But N is negative for $\rho > \rho_0$, so we must have $r > 1$ on this range, in particular also at $\rho = \rho_0$ where $N(\rho_0) = 0$ and so

$$\dot{N}(\rho_0) = 1 - \frac{(1 - w(\rho_0)^2)^2}{r(\rho_0)^2} > 0,$$

implying that $N > 0$ for sufficiently close $\rho \geq \rho_0$, which is a contradiction. $\square$

Lemma 5.3.13. *Consider a solution of the initial value problem (5.4a–5.4f, 5.8) with respect to fixed initial data in $\mathcal{I}_0$. Assume there is a point $\rho_0 \geq 0$ such that $N(\rho) > 0$ for all $\rho > \rho_0$.*

- (i) *If r is bounded, then the solution is asymptotically cylindrical, cf. Theorem 5.3.1 (ii).*
- (ii) *If r is unbounded, then the solution is asymptotically flat, cf. Theorem 5.3.1 (iii).*

Proof. Part (i) follows trivially from Lemma 5.3.10 since $N \rightarrow 0$ by Lemma 5.3.8 (ii). For part (ii), we first show that $U \rightarrow 0$. We consider separately the cases where w has finitely or infinitely many zeros.

If w has finitely many zeros, then there is a $\bar{\rho} \geq 0$ such that, without loss of generality, $0 < w(\rho) \leq 1$ for $\rho \geq \bar{\rho}$. Define the function

$$v(\rho) = \int_{\bar{\rho}}^{\rho} U,$$

so that $\dot{v} = U$ and $\ddot{v} = \dot{U}$. We will show that v has a limit at infinity. Since $\ddot{v} = \dot{U}$ is bounded by Lemma 5.3.8 (iii) and (5.4d), it will follow from Barbălat's lemma (Lemma 5.2.1) that $\dot{v} = U \rightarrow 0$. Integration by parts gives

$$v(\rho) = \int_{\bar{\rho}}^{\rho} \frac{\dot{w}}{r} = \frac{w(\rho)}{r(\rho)} - \frac{w(\bar{\rho})}{r(\bar{\rho})} + \int_{\bar{\rho}}^{\rho} \frac{w\dot{r}}{r^2}.$$

Since w is bounded and $r \rightarrow \infty$, it follows that the first term tends to 0, while the integral also has a finite limit since it is a bounded increasing function of ρ . Indeed, the integrand is positive and

$$\int_{\bar{\rho}}^{\rho} \frac{w\dot{r}}{r^2} \leq \sup_{\rho \geq \bar{\rho}} w(\rho) \int_{r(\bar{\rho})}^{r(\rho)} \frac{dr}{r^2} \leq \sup_{\rho \geq \bar{\rho}} w(\rho) \int_{r(\bar{\rho})}^{\infty} \frac{dr}{r^2} < \infty.$$

If w has infinitely many zeros, then so does U and we can find a sequence $\rho_k \rightarrow \infty$ with $U(\rho_k) = 0$. Since $\kappa - N \geq 0$, we see that the energy satisfies

$$\dot{E} = -4(\kappa - N)U^2 + \frac{N(1 - w^2)^2}{r^2} \leq \frac{N(1 - w^2)^2}{r^2},$$

and hence

$$E(\rho) - E(\rho_k) \leq \int_{\rho_k}^{\rho} \frac{N(1 - w^2)^2}{r^2} \leq \int_{\rho_k}^{\rho} \frac{\dot{r}}{r^3} = \frac{2}{r(\rho_k)^2} - \frac{2}{r(\rho)^2},$$

since $|w| \leq 1$ and $N > 0$ for $\rho \geq \rho_k$ if k is large enough. Thus, since $U(\rho_k) = 0$,

$$\begin{aligned} \limsup_{\rho \rightarrow \infty} E(\rho) &\leq E(\rho_k) + \frac{2}{r(\rho_k)^2} \\ &= 1 - \frac{(1 - w(\rho_k)^2)^2}{r(\rho_k)^2} + \frac{2}{r(\rho_k)^2} \rightarrow 1, \end{aligned}$$

where in the end we let $k \rightarrow \infty$. Since trivially $\liminf E \geq 1$ (because r is unbounded and w is bounded), we see that $E \rightarrow 1$. This implies that $2U^2 = E - 1 + \frac{(1 - w^2)^2}{r^2} \rightarrow 0$ in this case as well.

By Lemma 5.3.5, we now get $\kappa \rightarrow 1$. From the constraint (5.5), we see that

$$(1 - N)^2 = 1 - E + 2N(\kappa - 1) + \zeta^2 \rightarrow 0,$$

and hence $N \rightarrow 1$. Next, we wish to show that $\dot{w} \rightarrow 0$. Note that this does not follow directly from the fact that $U = \dot{w}/r \rightarrow 0$ since r is unbounded. It is not even clear, *prima*

facie, whether $\dot{w}$ is bounded - this does not follow from Lemma 5.3.8 (iii), since we do not consider $\dot{w}$ as one of the dependent variables.

Since $\kappa - 2N \rightarrow -1$ by the proof above, we may choose $\bar{\rho} \geq \rho_0$ so that $\kappa - 2N \leq -\frac{1}{2}$ for $\rho \geq \bar{\rho}$. The autonomous energy $F = 2\dot{w}^2 - (1 - w^2)^2$ satisfies

$$\dot{F} = -4(\kappa - 2N)\dot{w}^2 \geq 2\dot{w}^2 = F + (1 - w^2)^2 \geq F.$$

This implies, in particular, that the region $F \geq \varepsilon$ is invariant for any $\varepsilon > 0$. In this region, we have $2\dot{w}^2 = F + (1 - w^2)^2 \geq \varepsilon > 0$. Thus, if the orbit enters the region $F \geq \varepsilon$, then w exits the strip $|w| \leq 1$ at some finite ρ , which is a contradiction. Consequently, $F \leq 0$ for all $\rho \geq \bar{\rho}$, which also implies that $\dot{w}$ is bounded.⁷ Now

$$\int_{\bar{\rho}}^{\rho} \dot{w}^2 \leq -2 \int_{\bar{\rho}}^{\rho} (\kappa - 2N)\dot{w}^2 = \frac{1}{2} \int_{\bar{\rho}}^{\rho} \dot{F} \leq -\frac{1}{2} F(\bar{\rho})$$

and applying the monotone convergence theorem with $\rho \rightarrow \infty$ shows that $\dot{w}$ is square-integrable. Since $\ddot{w}$ is bounded, $\dot{w}^2$ is also uniformly continuous, and we get $\dot{w} \rightarrow 0$ by Barbălat's lemma (Lemma 5.2.1).

Now since F is non-decreasing for $\rho \geq \bar{\rho}$ and $F \leq 0$, we see that F has a limit at infinity. Hence, $(1 - w^2)^2 = 2\dot{w}^2 - F$ also has a limit at infinity, and consequently w also tends to some limit $|w_{\infty}| \leq 1$ by continuity. By (5.4c) we see that $\ddot{w} \rightarrow -w_{\infty}(1 - w_{\infty}^2)$ and consequently $w_{\infty} \in \{0, \pm 1\}$ (otherwise we could not have $\dot{w} \rightarrow 0$).

If $w_{\infty} = 0$, we need to show that the solution is the trivial $w \equiv 0$. In fact, we see that $F \rightarrow -1$, and since F is non-decreasing for large ρ and $F \geq -1$, we necessarily have $F \equiv -1$ which implies $w \equiv 0$. The discussion in Remark 5.3.2 also implies that in this case $r_0 > 1$.

Finally, we need to study the limits of $r\zeta$ and $r(1 - N^2)$ at infinity. Note that

$$\frac{d}{d\rho} \log(r\zeta) = N - \kappa \leq 0$$

by Lemma 5.3.4, so $r\zeta$ is non-increasing and

$$\lim_{\rho \rightarrow \infty} r\zeta = r_0 \sqrt{E_0} \exp \int_0^{\infty} (N - \kappa)$$

by the monotone convergence theorem. Therefore, to show that $r\zeta$ has a finite positive limit, it suffices to show that the integral on the right-hand side converges. By (5.6),

$$N - \kappa = -\frac{1}{N}(\dot{N} - 2U^2 + \zeta^2).$$

Hence, the claim will follow if we can show that U and ζ are square integrable near infinity, since $N \rightarrow 1$. But the integrability of U^2 follows trivially from the integrability

⁷This argument is similar in spirit to [BP92, Lemma 3–4].

of $\dot{w}^2 = (rU)^2$ (since $r \rightarrow \infty$), whereas the integrability of ζ^2 follows directly from the bound $\zeta^2 \leq E_0 \operatorname{sech}^2(\rho)$, cf. Lemma 5.3.4. On the other hand, we have

$$r(1 - N^2) = 2m = 2 \int_0^\rho \dot{m} = r_0 + \int_0^\rho \frac{\dot{r}}{r^2} [2\dot{w}^2 + (1 - w^2)^2 - (r\zeta)^2], \quad (5.14)$$

and the integral converges as $\rho \rightarrow 0$ since the quantity in the parentheses converges to the finite quantity $(1 - w_\infty^2)^2 - \alpha^2$. Therefore $r(1 - N^2)$ has a finite limit at infinity by dominated convergence, and this limit must be non-negative since $N \rightarrow 1$ and $N \leq 1$ (cf. Lemma 5.3.4). $\square$

We now have essentially all the ingredients necessary to complete the proof of Theorem 5.3.1. To summarize, the pieces of the puzzle can be put together as follows. In Lemma 5.3.6 we showed that the solution can be continued as long as N remains finite, and in particular the only way that a solution stops existing at a finite point $\rho = \rho_\infty$ is if $N \rightarrow \infty$ (and $r \rightarrow 0$), which is precisely the behaviour describing the singular orbits from Theorem 5.3.1 (i). In Lemma 5.3.7 we showed that such singular behaviour happens precisely when the solution enters the region $N + \zeta < 0$, and we also showed that the solution enters this region if it exits the strip $|w| \leq 1$. This allowed us to assume that the solution satisfies $N + \zeta \geq 0$ and $|w| \leq 1$, in which case it must be globally defined by Lemma 5.3.6 and the bounds from Lemma 5.3.4. In Lemma 5.3.8 we showed that the remaining dependent variables remain globally bounded and, most importantly, that r has a limit (which may be infinite) at infinity. The remainder of the proof then revolved around studying the different possible behaviours of N , depending on its number of zeros:

- ◊ In Lemma 5.3.11 we treat the case when N has infinitely many zeros, and show that the solution is then asymptotically cylindrical as in Theorem 5.3.1 (ii);
- ◊ In Lemma 5.3.12 we treat the case when N has finitely many zeros and is negative after the last zero, in which case the solution is singular as in Theorem 5.3.1 (i);
- ◊ In Lemma 5.3.13 we treat the final case when N has finitely many zeros and is positive after the last zero, in which case the solution asymptotically cylindrical Theorem 5.3.1 (ii) if r is bounded, and asymptotically flat as in Theorem 5.3.1 (iii) if r is unbounded.

Since this exhausts all the possible cases, the proof of Theorem 5.3.1 is complete.

5.4 Neighbourhood theorem

In this section, we will study the set of admissible initial data

$$\mathcal{I}_0 = \{(r_0, w_0, U_0) \in \mathbb{R}^3 \mid r_0 > 0, |w_0| \leq 1, E_0 \geq 0\},$$

and particularly its subsets generated by the classification in Theorem 5.3.1. We will hereafter fix $r_0 > 0$, and a given subset $\mathcal{Y} \subset \mathcal{I}_0$ we will write

$$\mathcal{Y}(r_0) = \{(w_0, U_0) \in \mathbb{R}^2 \mid (r_0, w_0, U_0) \in \mathcal{Y}\}.$$

We also introduce the following notation, which slightly differs from the categorization in Theorem 5.3.1, but it will turn out to be more convenient throughout this section.

Definition 5.4.1. We define the following subsets of $\mathcal{I}_0$:

- ◊ For the *singular orbits*, i.e. those with $r \rightarrow 0$ and $N \rightarrow -\infty$ at some finite ρ , we define the subsets (denoting by n the number of zeros of w for $\rho > 0$):
 - ◊ $\mathcal{E}_n$ as the set of *escaping singular orbits*, for which w escapes the strip $|w| \leq 1$.
 - ◊ $\mathcal{C}_n$ as the set of *crashing singular orbits*, for which w stays in $|w| \leq 1$. We also put $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$.
- ◊ $\mathcal{O}$ as the set of *oscillatory orbits*, defined for all $\rho > 0$ with $(w, \dot{w}) \rightarrow (0, 0)$.
- ◊ $\mathcal{R}_n$ as the set of *regular orbits*, defined for all $\rho > 0$ with $(|w|, \dot{w}) \rightarrow (1, 0)$, and such that w has n zeros for $\rho > 0$.

By Theorem 5.3.1, we see that the sets $\mathcal{R}_n, \mathcal{E}_n, \mathcal{C}, \mathcal{O}$ form a disjoint partition of $\mathcal{I}_0$. We also observe that all orbits in $\mathcal{R}_n$ are asymptotically flat. For orbits in $\mathcal{O}(r_0)$, we have that:

- ◊ if $r_0 > 1$, then $w \equiv 0$ and the orbit is asymptotically flat,
- ◊ if $r_0 = 1$, then $w \equiv 0$ and the orbit is asymptotically cylindrical,
- ◊ if $r_0 < 1$, then $w \not\equiv 0$ and the orbit is asymptotically cylindrical.

Finally, we note that $\mathcal{C}(r_0) = \emptyset$ for $r_0 \geq 1$, cf. Lemma 5.3.9.

Our current goal is to show the following result, which tells us how the orbits near a given orbit type behave.

Theorem 5.4.2 (Neighbourhood theorem). *Fix $r_0 > 0$ and let $x_0 = (w_0, U_0) \in \mathcal{I}_0(r_0)$. Define the slices*

$$\begin{aligned}\mathcal{W}_\delta(r_0, x_0) &= \{(\lambda, U_0) \in \mathcal{I}_0(r_0) \mid |\lambda - w_0| \leq \delta\}, \\ \mathcal{U}_\delta(r_0, x_0) &= \{(w_0, \mu) \in \mathcal{I}_0(r_0) \mid |\mu - U_0| \leq \delta\}.\end{aligned}$$

(i) *Let either*

$$\mathcal{Y}(r_0) = \mathcal{E}_n(r_0) \quad \text{or} \quad \mathcal{Y} = \mathcal{C}(r_0) \setminus \{0\}.$$

If $x_0 \in \mathcal{Y}(r_0)$ and $\delta > 0$ is sufficiently small, we have

$$\mathcal{U}_\delta(r_0, x_0) \subset \mathcal{Y}(r_0).$$

If we also have $w_0 \neq 0$, then

$$\mathcal{W}_\delta(r_0, x_0) \subset \mathcal{Y}(r_0).$$

(ii) *If $x_0 \in \mathcal{O}(r_0)$, then for any $n \geq 0$ we can choose $\delta > 0$ so small that the y_0 -orbit has at least n zeros of w for $\rho > 0$, whenever $0 < |y_0 - x_0| \leq \delta$.*

(iii) If $x_0 \in \mathcal{R}_n(r_0) \setminus \{0\}$ and $\delta > 0$ is sufficiently small, we have

$$\mathcal{U}_\delta(r_0, x_0) \subset (\mathcal{R}_n \cup \mathcal{E}_n \cup \mathcal{E}_{n+1})(r_0).$$

If we also have $w_0 \neq 0$, then

$$\mathcal{W}_\delta(r_0, x_0) \subset (\mathcal{R}_n \cup \mathcal{E}_n \cup \mathcal{E}_{n+1})(r_0).$$

Remark 5.4.3. This result is an analogue of [BFM94, Propositions 30 and 32] for the wormhole setting. Part (i) essentially says that $\mathcal{C}(r_0)$ and each $\mathcal{E}_n(r_0)$ are open relative to the lines where either w_0 is fixed, or U_0 is fixed and $w_0 \neq 0$. In parts (ii) and (iii), one can actually get a stronger result. In fact, one can show that oscillatory and regular orbits are locally unique (cf. [BFM94, Propositions 31 and 33]). This fact is, however, non-essential for the proofs in §5.6, nor does it improve the statements of the corresponding theorems, so we omit them.⁸ Finally, we would like to point out that the proof of part (iii) given in [BFM94, Lemma 20] for their setting again invokes the theory of structurally stable vector fields, which we have already commented on in Remark 5.2.2. The proof given here naturally avoids this.

Remark 5.4.4. Throughout the proofs in this section, we will repeatedly use some basic facts about ordinary differential equations. As we have already observed in §5.1, the solution depends continuously on the initial data $(w_0, U_0) \in \mathcal{I}_0(r_0)$. Furthermore, classical results [Har02, Theorem 3.2] ensure that the maximal forward point of existence $\bar{\rho} > 0$ of the solution is a lower-semicontinuous function of the initial data. This is important to us because we would like to compare the values of an orbit with its nearby orbits near the end of its existence.

Consider now an orbit with initial data $x_0 = (w_0, U_0) \in \mathcal{I}_0$. The x_0 -orbit is either singular and hence defined up to some finite ρ_∞ , or it exists for all $\rho \geq 0$. As already noted, we can choose $\delta > 0$ so that the solutions with initial data y_0 such that $|y_0 - x_0| \leq \delta$ are defined up to any $\bar{\rho} < \rho_\infty$ in the former case, or up to $\bar{\rho}$ as large as we would like in the latter case.

By further shrinking $\delta > 0$ if necessary, we can also ensure that the values of the dependent variables at $\bar{\rho}$ differ by no more than any given $\varepsilon > 0$, since the solutions depend continuously on the initial parameters. Moreover, if the x_0 -orbit is defined for all $\rho \geq 0$ and a continuous function of its dependent variables, call it ξ , tends to some limit L , then we can for any $\varepsilon, K > 0$ find $\bar{\rho} > 0$ (depending only on ε) so large and $\delta > 0$ so small that $\xi(\rho, y_0)$ differs from L by no more than ε on the interval $\bar{\rho} \leq \rho \leq \bar{\rho} + K$. Indeed, we can first choose $\bar{\rho}$ so large that $|\xi(\rho, x_0) - L| \leq \varepsilon/2$ for $\rho \geq \bar{\rho}$, and δ small enough that orbits with initial data in $\{|y_0 - x_0| \leq \delta\}$ are defined at least up to $\bar{\rho} + K$. Then ξ is continuous on the compact set

$$[\bar{\rho}, \bar{\rho} + K] \times \{|y_0 - x_0| \leq \delta\},$$

⁸The authors of the cited paper likely included this analysis as it seems like they were also attempting to get some global uniqueness result for regular orbits with a given number of zeros of w .

so it is also uniformly continuous there. By shrinking δ further if necessary we then get $|\xi(\rho, y_0) - \xi(\rho, x_0)| \leq \varepsilon/2$ for all $(\rho, y_0) \in [\bar{\rho}, \bar{\rho} + K] \times \{|y_0 - x_0| \leq \delta\}$, which gives the desired claim.

Finally, we note that if $w(\rho, x_0)$ has (exactly) n zeros for $0 < \rho < \bar{\rho}$ and $w(\bar{\rho}, x_0) \neq 0$ (in particular $w \not\equiv 0$), then for sufficiently small δ and initial data y_0 with either $y_0 \in \mathcal{U}_\delta(r_0, x_0)$, or $y_0 \in \mathcal{W}_\delta(r_0, x_0)$ with $w_0 \neq 0$, the corresponding solution $w(\rho, y_0)$ has (exactly) n zeros for $0 < \rho < \bar{\rho}$. To see this, let $\{\rho_k \mid k = 1, \dots, n\}$ be the n zeros of $w(\rho, x_0)$ on $0 < \rho < \bar{\rho}$. If $w_0 = 0$ we also put $\rho_0 = 0$ and let k run from 0 to n instead. Then $\dot{w}(\rho_k, x_0) \neq 0$ since otherwise we would have $w \equiv 0$ (cf. Remark 5.1.1), and we can by continuity find $\delta, \gamma > 0$ so small that

$$\dot{w}(\rho, y_0) \neq 0, \quad \rho \in (\rho_k - \gamma, \rho_k + \gamma) =: I_k,$$

for each k and all y_0 -orbits (note that γ can be chosen uniformly because k runs over a finite set). In particular, $w(\rho, y_0)$ is strictly monotone in each I_k . On the other hand, $\rho \mapsto |w(\rho, x_0)|$ is positive on the compact set

$$K = [0, \bar{\rho}] \setminus \bigcup_k I_k,$$

so it also has a positive minimum on K and we may select $\varepsilon > 0$ strictly smaller than this minimum. Then, finally, we shrink δ further if necessary to make

$$|w(\rho, y_0) - w(\rho, x_0)| < \varepsilon,$$

which also ensures that, for each fixed y_0 , the corresponding $\rho \mapsto w(\rho, y_0)$ has no zeros on K , and has exactly one zero in each I_k by monotonicity and the intermediate value theorem (note that the zeros are no longer necessarily located at ρ_k for the y_0 -orbits). This gives the claim if $w_0 \neq 0$. On the other hand, if $w_0 = 0$, then we note that the zero at $\rho_0 = 0$ stays located at $\rho = 0$ for the nearby y_0 -orbits, since we only consider $y_0 \in \mathcal{U}_\delta(r_0, x_0)$ in this case, and consequently the number of zeros for $\rho > 0$ remains unaffected.⁹

Proof of Theorem 5.4.2. Assume first that $x_0 \in \mathcal{E}_n(r_0) \setminus \{0\}$ and choose $\delta > 0$ so small and $\bar{\rho}$ so close to the singular point that $(N + \zeta)(\bar{\rho}, y_0) < 0$ and $w(\bar{\rho}, y_0) > 1$, and w has exactly n zeros for the desired initial data $y_0 \in \mathcal{U}_\delta(r_0, x_0)$ or $y_0 \in \mathcal{W}_\delta(r_0, x_0)$ with $w_0 \neq 0$ (cf. Remark 5.4.4). All of the y_0 -orbits are then singular by Lemma 5.3.7, and $w(\rho, y_0)$ cannot gain any more zeros for $\rho > \bar{\rho}$ because it stays in the region $|w| > 1$. It follows that $y_0 \in \mathcal{E}_n(r_0)$.

An analogous argument can be applied when $x_0 \in \mathcal{C}(r_0) \setminus \{0\}$ to conclude that the nearby y_0 -orbits are singular (but in this case $w(\rho, y_0)$ could gain zeros for $\rho > \bar{\rho}$, which

⁹This is important in view of the way we count zeros of w (cf. Definition 5.4.1). It is also the main reason for not allowing the case $w_0 = 0$ when considering the slices $\mathcal{W}_\delta$ in the w_0 -direction. Indeed, in this case the initial zero of w could move around too much, which would also destroy the counting of the zeros near $\rho = 0$.

is why we refrain from counting them in the first place). Furthermore, the energy (5.11) is non-increasing for $N \leq -\zeta$ and satisfies

$$E = 2\kappa N - N^2 + \zeta^2 \leq 0, \quad \text{i.e.} \quad \frac{(1 - w^2)^2}{r^2} \geq 1 + 2U^2 \geq 1, \quad (5.15)$$

so w cannot cross the lines $|w| = 1$ in this region and consequently $y_0 \in \mathcal{C}(r_0) \setminus \{0\}$.

Now consider the oscillatory case (ii) of Theorem 5.4.2. If $r_0 < 1$, the x_0 -orbit has infinitely many zeros of w while $w \not\equiv 0$, so the result follows trivially by continuity with respect to initial data, cf. Remark 5.4.4 (alternatively, one can use a similar argument as in the case $r_0 = 1$ below, if one prefers). For $r_0 \geq 1$, we first note that the orbits with initial data y_0 in the punctured ball $0 < |y_0 - x_0| \leq \delta$ cannot be in $\mathcal{O}(r_0)$ or $\mathcal{C}(r_0)$, as follows from Theorem 5.3.1 and the fact that no orbit can crash for $r_0 \geq 1$, cf. Lemma 5.3.9. To show that these have arbitrarily many zeros, we consider the polar angle of $(w, \dot{w})$ defined by

$$-2\pi < \theta(0) \leq 0, \quad \tan \theta = \frac{\dot{w}}{w} \quad \text{if } w \neq 0,$$

and extended smoothly across zeros of w . Note that this is well-defined since the orbits starting in the punctured ball $0 < |y_0 - x_0| \leq \delta$ cannot reach the fixed point $(w, \dot{w}) = (0, 0)$. A simple calculation yields (cf. (5.13))

$$\dot{\theta} + \frac{1}{2} \leq \frac{1}{2} |\kappa - 2N \pm 1| + w^2.$$

The idea is to show that the right-hand side is sufficiently small on an arbitrarily large interval, which will imply arbitrarily many zeros of w . Note that:

- ◊ If $r_0 = 1$, then necessarily $x_0 = (1, 0, 0)$ and $w \equiv 0$, $N \equiv 0$, $\kappa = \tanh(\rho) \rightarrow 1$.
- ◊ If $r_0 > 1$, then necessarily $x_0 = (r_0, 0, 0)$ and $w \equiv 0$, while $\kappa \rightarrow 1$ and $N \rightarrow 1$ since these orbits are asymptotically flat, cf. Remark 5.3.2.

Hence, in both cases we can for any $K > 0$ find $\bar{\rho} > 0$ sufficiently large (independent of K) and $\delta > 0$ so small that (cf. Remark 5.4.4)

$$|(\kappa - 2N)(\rho, y_0) \pm 1| \leq \frac{1}{4} \quad \text{and} \quad w(\rho, y_0)^2 \leq \frac{1}{8},$$

for $\bar{\rho} \leq \rho \leq \bar{\rho} + K$ and all y_0 -orbits with $|y_0 - x_0| \leq \delta$, where the first inequality holds with the minus sign for $r_0 = 1$, and with the plus sign for $r_0 > 1$. It therefore follows that $\dot{\theta}(\rho, y_0) \leq -\frac{1}{4}$ for $\bar{\rho} \leq \rho \leq \bar{\rho} + K$ and $|y_0 - x_0| \leq \delta$, implying that

$$\theta(\bar{\rho} + K, y_0) = \theta(\bar{\rho}, y_0) + \int_{\bar{\rho}}^{\bar{\rho}+K} \dot{\theta}(\cdot, y_0) \leq \theta(\bar{\rho}, y_0) - \frac{K}{4}$$

But the right-hand side can be made arbitrarily (negatively) large, which implies arbitrarily many zeros of w .

Finally, we consider part (iii), so let $x_0 \in \mathcal{R}_n(r_0) \setminus \{0\}$. We first want to make sure that, for nearby orbits, N and κ stay near 1. Since E_0 is bounded on $|y_0 - x_0| \leq \delta$, we see that for any given $\bar{\varepsilon} \leq 1$ and sufficiently large $\bar{\rho}$ depending on it, all orbits with initial data in $|y_0 - x_0| \leq \delta$ have

$$\sqrt{E_0(y_0)} \operatorname{sech}(\rho) \leq 2\bar{\varepsilon} \quad \text{and} \quad (\kappa + N)(\rho, y_0) \leq 2(1 + \bar{\varepsilon})$$

for all $\rho \geq \bar{\rho}$, where the latter inequality follows by the former and Lemma 5.3.4. Since the x_0 -orbit is asymptotically flat, it has $r \rightarrow \infty$ and $N \rightarrow 1$, so we may further increase $\bar{\rho}$ and shrink δ if necessary to ensure that

$$N(\bar{\rho}, y_0) \geq 1 - \bar{\varepsilon}, \quad r(\bar{\rho}, y_0) \geq \frac{1}{\bar{\varepsilon}}$$

for all orbits with initial data in $|y_0 - x_0| \leq \delta$. As long as $N \geq 0$ and $|w| \leq 1$, we then have $r \geq r(\bar{\rho}, y_0) \geq \frac{1}{\bar{\varepsilon}}$, and equation (5.4b) gives

$$\dot{N} \geq 1 - \bar{\varepsilon}^2 - [2(1 + \bar{\varepsilon}) - N]N \geq (1 + \bar{\varepsilon} - N)^2 - 4\bar{\varepsilon},$$

which shows that N increases in the region $N < 1 + \bar{\varepsilon} - 2\sqrt{\bar{\varepsilon}}$ and in particular stays positive if $\bar{\varepsilon}$ is sufficiently small. There are two options, as long as $|w| \leq 1$:

- ◊ If $N(\bar{\rho}, y_0) < 1 + \bar{\varepsilon} - 2\sqrt{\bar{\varepsilon}}$, then $N(\rho, y_0) \geq N(\bar{\rho}, y_0) \geq 1 - \bar{\varepsilon}$ for $\rho \geq \bar{\rho}$.
- ◊ If $N(\bar{\rho}, y_0) \geq 1 + \bar{\varepsilon} - 2\sqrt{\bar{\varepsilon}}$, then $N(\rho, y_0) \geq 1 + \bar{\varepsilon} - 2\sqrt{\bar{\varepsilon}}$ for $\rho \geq \bar{\rho}$.

Hence, for any $\varepsilon > 0$, we can choose $\bar{\varepsilon}$ small enough that

$$N(\rho, y_0) \geq 1 - \max\{\bar{\varepsilon}, 2\sqrt{\bar{\varepsilon}} - \bar{\varepsilon}\} \geq 1 - \varepsilon$$

$$\kappa(\rho, y_0) \leq 2 + \bar{\varepsilon} - N(\rho, y_0) \leq 1 + \bar{\varepsilon} + \max\{\bar{\varepsilon}, 2\sqrt{\bar{\varepsilon}} - \bar{\varepsilon}\} \leq 1 + \varepsilon$$

for all $\rho \geq \bar{\rho}$ as long as $|w| \leq 1$, and all orbits with initial data y_0 in $|y_0 - x_0| \leq \delta$, as desired.

Since the x_0 -orbit has exactly n zeros of w and $|w| \rightarrow 1$, we can again shrink δ and increase $\bar{\rho}$ to ensure that the y_0 -orbits such that either $y_0 \in \mathcal{U}_\delta(r_0, x_0)$, or $y_0 \in \mathcal{W}_\delta(r_0, x_0)$ with $w_0 \neq 0$, satisfy $(1 - w(\bar{\rho}, y_0))^2 \leq \varepsilon$ and have exactly n zeros of w for $0 < \rho < \bar{\rho}$, cf. Remark 5.4.4. The autonomous energy (5.12) satisfies

$$\dot{F} = -4(\kappa - 2N)\dot{w}^2 \geq 4(1 - 3\varepsilon)\dot{w}^2.$$

In particular, this shows that F is non-decreasing for $\varepsilon < 1/3$, so $F(\rho, y_0) \geq F(\bar{\rho}, y_0) \geq -\varepsilon$, or equivalently

$$2\dot{w}^2 \geq (1 - w^2)^2 - \varepsilon, \tag{5.16}$$

for $\rho \geq \bar{\rho}$ with $|w| \leq 1$. Now there are three options for the y_0 -orbit (we omit y_0 as an argument henceforth), where we assume without loss of generality that w and $\dot{w}$ are negative at $\bar{\rho}$, in view of the symmetry $(w, U) \mapsto -(w, U)$:

- (i) w continues decreasing but stays in the region $|w| \leq 1$, hence the orbit is in $\mathcal{R}_n(r_0)$,
- (ii) w continues decreasing and enters the region $w < -1$, hence the orbit is in $\mathcal{E}_n(r_0)$,
- (iii) w decreases until it reaches a minimum at some point $\rho_1 \geq \bar{\rho}$, so $\dot{w}(\rho_1) = 0$ and $-1 < w(\rho_1) < 0$, and then turns back towards the region $w > 0$.

To complete the proof of (iii), we thus need to show that w enters the region $w > 1$ in case (iii), so the orbit is in $\mathcal{E}_{n+1}(r_0)$. The argument below is, in essence, very similar to [SW93, Proposition 4.8]. Define $b > a \geq \rho_1$ as the first points such that $w(a) = -\sqrt{1 - \sqrt{\varepsilon}}$ and $w(b) = 0$. These are well-defined because $\dot{w}$ increases and $\dot{w} > 0$ in the region $w < 0$ for $\rho > \rho_1$ by (5.4c), so w also increases uniformly in that region. Then for $a \leq \rho \leq b$, the right-hand side of (5.16) is non-negative, so we may take the square root and estimate

$$\begin{aligned}
F(\rho) &\geq -\varepsilon + 4(1 - 3\varepsilon) \int_{\rho_1}^{\rho} \dot{w}^2 \\
&\geq -\varepsilon + 2\sqrt{2}(1 - 3\varepsilon) \int_a^b \dot{w} \sqrt{(1 - w^2)^2 - \varepsilon} \\
&= -\varepsilon + 2\sqrt{2}(1 - 3\varepsilon) \int_{-\sqrt{1 - \sqrt{\varepsilon}}}^0 \sqrt{(1 - w^2)^2 - \varepsilon} dw,
\end{aligned}$$

for $\rho \geq b$. The right-hand side depends continuously on ε and tends to $\frac{4\sqrt{2}}{3}$ as $\varepsilon \rightarrow 0$. This implies that F is lower bounded by a positive constant for $\rho \geq b$ when ε is sufficiently small. Hence, the same is true for $\dot{w}$, implying that w keeps increasing uniformly and eventually enters the region $w > 1$, as desired. $\square$

Remark. We would like to point out that in [SWY93, Proposition 3.5], the authors prove the analogue of part (ii), $r_0 > 1$, of Theorem 5.4.2 for their setting. However, their proof is much more involved, because they treat r as the independent variable, so they cannot use continuous dependence on initial data since the initial point for the x_0 -orbit is a singular point of the system in those coordinates. However, this singularity only appears as a consequence of using r as a coordinate, and in our system we do not face the same difficulties, so the proof is much simpler.

5.5 Energy theorem

Having studied neighbourhoods of different types of orbits, we also wish to understand how the extremal parts of the set $\mathcal{S}_0$ look like. This can be viewed as a compactness result, as it will provide us with appropriate upper and lower bounds for the shooting method.

Theorem 5.5.1 (Energy theorem). *Fix $r_0 > 0$, let $(w_0, U_0) \in \mathcal{S}_0(r_0)$ and*

$$E_0 = E_0(w_0, U_0) = 1 + 2U_0^2 - \frac{(1 - w_0^2)^2}{r_0^2}.$$

- (i) If $E_0 = 0$, then either $(r_0, w_0, U_0) = (1, 0, 0) \in \mathcal{O}$ or $(w_0, U_0) \in \mathcal{C}(r_0)$ with $r_0 < 1$.
- (ii) If E_0 is sufficiently large, then w is monotone and $(w_0, U_0) \in (\mathcal{E}_0 \cup \mathcal{E}_1)(r_0)$.

Remark. In part (ii), the required magnitude of E_0 depends on r_0 . The heuristic idea behind part (ii) is that if the initial velocity $\dot{w}_0$ of w is chosen large enough (this is in fact equivalent to choosing the initial energy E_0 large enough since r_0 is fixed), then w escapes the strip $|w| \leq 1$. Even though this seems intuitively obvious, the proof is surprisingly difficult. The first issue is that, a priori, the orbit could crash arbitrarily fast for large E_0 , before w escapes the strip $|w| \leq 1$. The second difficulty is that, even if the orbit does not crash, equation (5.4d) implies that the larger E_0 is, the faster U decreases, so it could become negative arbitrarily fast, in particular if $\kappa - N$ becomes large. The crux of the issue here is that we do not have a bound for κ that is uniform in E_0 (in particular, the bound from Lemma 5.3.4 is not good enough).

Proof of Theorem 5.5.1 (i). Let $E_0 = 0$ and recall that $\zeta_0 = \sqrt{E_0} = 0$. By (5.4f), we then see that $\zeta \equiv 0$. In view of this, Lemma 5.3.7 implies that the region $N < 0$ is invariant and the orbit is singular if it enters it.

In fact, we see from (5.6) that $\dot{N}_0 = -2U_0^2$, which is negative if $U_0 \neq 0$, so that $N < 0$ for $\rho > 0$ in this case. On the other hand, if $U_0 = 0$, then $\dot{N}_0 = \ddot{N}_0 = 0$, but

$$\ddot{N}_0 = -\frac{4w_0^2(1 - w_0^2)}{r_0^2},$$

so that either $N < 0$ for $\rho > 0$, or else $(r_0, w_0, U_0) = (1, 0, 0) \in \mathcal{O}$, cf. Remark 5.3.2.

This shows that the non-trivial solutions with $E_0 = 0$ are singular, so it only remains to show that they crash rather than escape the strip $|w| \leq 1$. But this follows immediately since $|w_0| < 1$ (note that $E_0 = 0$ prohibits $|w_0| = 1$), and w cannot cross the lines $|w| = 1$ in the region $N \leq -\zeta \equiv 0$, cf. (5.15). $\square$

As already hinted, the proof of Theorem 5.5.1 (ii) is much more involved. We will need the following lemma, which allows us to control r uniformly for all choices of initial parameters with fixed r_0 .

Lemma 5.5.2. *Fix $r_0 > 0$. For each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any solution of the initial value problem (5.4a–5.4f, 5.8) with respect to initial data $x_0 = (w_0, U_0) \in \mathcal{J}_0(r_0)$, we have*

$$|r(\rho, x_0) - r_0| \leq \varepsilon,$$

for all $0 \leq \rho \leq \delta$ for which $|w| \leq 1$.

Remark. The main point of the result is that δ can be chosen independently of (w_0, U_0) .

Proof. By Lemma 5.3.4, we know that $N \leq 1$, so we only need a lower bound that is uniform for $(w_0, U_0) \in \mathcal{J}_0(r_0)$. To this end, we compare the equations (5.4a, 5.4b)

for (r, N) with a simpler system. For $r_0 < 1$, consider the solution of the initial value problem¹⁰

$$\begin{cases} \dot{\bar{r}} = \bar{r}\bar{N}, & \bar{r}(0) = r_0, \\ \dot{\bar{N}} = 1 - \frac{1}{\bar{r}^2}, & \bar{N}(0) = 0, \end{cases} \quad (5.17)$$

and define

$$(\bar{r}, \bar{N}) = \begin{cases} (r_0, 0) & \text{if } r_0 \geq 1, \\ \text{solution of (5.17)} & \text{if } r_0 < 1. \end{cases}$$

We claim that for any $x_0 = (w_0, U_0) \in \mathcal{J}_0(r_0)$, we have

$$r(\rho, x_0) \geq \bar{r}(\rho) \quad \text{and} \quad N(\rho, x_0) \geq \bar{N}(\rho)$$

for all ρ for which $(\bar{r}, \bar{N})$ is well-defined and $|w(\rho, x_0)| \leq 1$.

If $r_0 \geq 1$, then the region $N \geq 0$ is invariant (cf. Lemma 5.3.9) and the result follows trivially. So assume $r_0 < 1$ and take any $(w_0, U_0) \in \mathcal{J}_0(r_0)$. Define

$$\xi = r - \bar{r} \quad \text{and} \quad \eta = N - \bar{N}.$$

Then $\xi(0) = \eta(0) = 0$ and using (5.4a, 5.17) we can also calculate

$$\xi = r_0 \left(\exp \int_0^\rho N - \exp \int_0^\rho \bar{N} \right) = r_0 \bar{r} \left(\exp \int_0^\rho \eta - 1 \right).$$

This shows that $\xi > 0$ (at least) as long as $\eta > 0$. We first wish to show that $\eta > 0$ at least for sufficiently small $\rho > 0$. To this end, we have by (5.4b, 5.17)

$$\dot{\eta}(0) = \frac{1 - (1 - w_0^2)^2}{r_0^2},$$

so that $\dot{\eta}(0) > 0$ if $|w_0| \neq 0$. If $w_0 = 0$, then $\dot{\eta}(0) = 0$ gives no information, so we continue differentiating (5.4b, 5.17) to find $\ddot{\eta}(0) = 0$ but

$$\ddot{\eta}(0) = 4U_0^2 - 2(1 + 2U_0^2) \left(1 - \frac{1}{r_0^2} \right) \geq 2 \left(\frac{1}{r_0^2} - 1 \right) > 0,$$

since $r_0 < 1$. Thus η increases initially in both cases and thus $\eta > 0$ for sufficiently small $\rho > 0$.

Suppose now that η ever reaches zero again, so that there exists a point $\bar{\rho} > 0$ with $\eta(\bar{\rho}) = 0$ and $\eta > 0$ for $0 < \rho < \bar{\rho}$. This implies that $N(\bar{\rho}) = \bar{N}(\bar{\rho}) < 0$, and by (5.4b,

¹⁰It appears that (5.17) does not have an explicit solution, but we would like to note that the solution satisfies $\bar{N} < 0$ for $\rho > 0$ until it stops existing at $\bar{r} = 0$.

5.17), $\dot{\eta}$ satisfies

$$\begin{aligned}\dot{\eta}(\bar{\rho}) &= \left(\frac{1}{\bar{r}^2} - \frac{(1-w^2)^2}{r^2} - \kappa N \right) \Big|_{\bar{\rho}} \\ &\geq \frac{r + \bar{r}}{(r\bar{r})^2} \Big|_{\bar{\rho}} \xi(\bar{\rho}) \\ &= \frac{r_0 \bar{r}(r + \bar{r})}{(r\bar{r})^2} \Big|_{\bar{\rho}} \left(\exp \int_0^{\bar{\rho}} \eta - 1 \right),\end{aligned}$$

where we also use the fact that $|w| \leq 1$ and $\kappa \geq 0$ (cf. Lemma 5.3.4). But since $\eta > 0$ for $0 < \rho < \bar{\rho}$, we see that the latter is positive, so such a point cannot exist and $\eta > 0$ for all $\rho > 0$. Thus also $\xi > 0$ and in particular $r > \bar{r}$ and $N > \bar{N}$ for $\rho > 0$.

We thus have $\bar{N} \leq N \leq 1$, and since $\bar{N}$ is a fixed function independent of the initial data $(w_0, U_0) \in \mathcal{J}_0(r_0)$ (but depending on r_0), Lemma 5.5.2 follows immediately. $\square$

Proof of Theorem 5.5.1 (ii). Fix $r_0 > 0$ and use Lemma 5.5.2 to find a $\delta > 0$ so that

$$\frac{1}{2} \leq \frac{r}{r_0} \leq 2, \quad (5.18)$$

as long as $|w| \leq 1$, for any orbit with initial data $(w_0, U_0) \in \mathcal{J}_0(r_0)$. For later purposes we also shrink δ if necessary, to ensure that¹¹

$$\delta < \min \left\{ 1, \frac{r_0}{2(r_0 + 16)} \log \left(1 + \frac{r_0 + 16}{2^{13} 3 r_0} \right) \right\}. \quad (5.19)$$

Now define

$$\bar{\rho} = \bar{\rho}(\delta, w_0, U_0) = \sup \{ 0 \leq s \leq \delta \mid |w(\rho)| \leq 1 \text{ for } 0 \leq \rho \leq s \},$$

so that the bounds (5.18) hold for $0 \leq \rho \leq \bar{\rho}$. We will show that w is strictly monotone for $0 \leq \rho \leq \bar{\rho}$ and that $\bar{\rho} < \delta$ for sufficiently large E_0 , which implies the desired result.

In view of the symmetry $(w, U) \mapsto -(w, U)$ and the fact that large E_0 necessitates large U_0^2 , we can without loss of generality assume that U_0 is positive (on the other hand, w_0 could be negative, but satisfies $|w_0| \leq 1$). We begin by observing that (5.4c) can equivalently be written as

$$\frac{d}{d\rho} \left(\frac{U}{r\zeta} \right) = - \frac{w(1-w^2)}{r^2\zeta},$$

and an integration gives

$$U = \frac{U_0}{\zeta_0} \frac{r}{r_0} \zeta - r\zeta \int_0^\rho \frac{w(1-w^2)}{r^2\zeta}. \quad (5.20)$$

¹¹Though these assumptions seem arbitrary at the moment, particularly the latter one, their significance will become apparent as we go along.

Observe that $E_0 \rightarrow \infty$ if and only if $U_0 \rightarrow \infty$ (since r_0 is fixed and $|w_0| \leq 1$), and

$$\frac{U_0}{\zeta_0} = \frac{U_0}{\sqrt{E_0}} \rightarrow \frac{1}{\sqrt{2}} \quad \text{as } E_0 \rightarrow \infty,$$

so for sufficiently large E_0 , we have $\frac{1}{2} \leq U_0/\zeta_0 \leq 1$. Thus, using also the bounds (5.18), we can estimate the first term in (5.20) as

$$\frac{1}{4} \zeta \leq \frac{U_0}{\zeta_0} \frac{r}{r_0} \zeta \leq 2\zeta,$$

for $0 \leq \rho \leq \bar{\rho}$. For the second term in (5.20), we note that $r\dot{\zeta}$ does not increase, since

$$\frac{d}{d\rho} \log(r\dot{\zeta}) = N - \kappa \leq 0$$

by Lemma 5.3.4. Since $|w| \leq 1$ for $0 \leq \rho \leq \bar{\rho}$, we get

$$\left| r\dot{\zeta} \int_0^\rho \frac{w(1-w^2)}{r^2\dot{\zeta}} \right| \leq r\dot{\zeta} \int_0^\rho \frac{1}{r^2\dot{\zeta}} \leq \int_0^\rho \frac{1}{r} \leq \frac{2}{r_0},$$

where we use the bounds (5.18) and also the assumption $\rho \leq \delta < 1$. It follows that, for $0 \leq \rho \leq \bar{\rho}$,

$$\frac{1}{4} \zeta - \frac{2}{r_0} \leq U \leq 2 \left(\zeta + \frac{1}{r_0} \right),$$

This then gives (since $\dot{w} = rU$)

$$\dot{w} \geq \frac{r_0}{8} \zeta - 4 \quad \text{and} \quad 2U^2 \leq 16 \left(\zeta^2 + \frac{1}{r_0} \right), \quad (5.21)$$

where we also use the estimate $(x+y)^2 \leq 2(x^2+y^2)$ to achieve the second inequality and again the bounds (5.18) for the first one. If we define $\phi(\rho) = \int_0^\rho \zeta$, then we see by integrating the first inequality in (5.21) that

$$\phi \leq \frac{8}{r_0} (w - w_0 + 4\rho) \leq \frac{48}{r_0} \quad \text{for } 0 \leq \rho \leq \bar{\rho}, \quad (5.22)$$

since $|w| \leq 1$ and $\rho \leq \delta < 1$. On the other hand, since $\dot{\zeta} = -\kappa\zeta$ by (5.4f), we get

$$\frac{d}{d\rho} \left(\frac{\kappa}{\zeta} \right) = \frac{1+2U^2}{\zeta} \leq \left(1 + \frac{16}{r_0} \right) \frac{1}{\zeta} + 16\zeta,$$

where we also use the second inequality from (5.21). Since $\kappa(0) = 0$, an integration gives

$$\begin{aligned} \kappa &\leq \left(1 + \frac{16}{r_0} \right) \zeta \int_0^\rho \frac{1}{\zeta} + 16\zeta \int_0^\rho \zeta \\ &\leq \left(1 + \frac{16}{r_0} \right) \rho + 16\zeta\phi \\ &\leq b + a\zeta, \end{aligned}$$

where the second step follows since ζ decreases, whereas in the last step we use (5.22) to obtain the positive constants $a = \frac{2^{83}}{r_0}$, $b = 1 + \frac{16}{r_0}$, both depending only on the fixed r_0 . Using $\dot{\zeta} = -\kappa\zeta$ again, we may then integrate to get

$$\begin{aligned}\rho &\geq \int_0^\rho \frac{\kappa}{b + a\zeta} = - \int_{\sqrt{E_0}}^{\zeta(\rho)} \frac{d\zeta}{\zeta(b + a\zeta)} \\ &= \frac{1}{b} \log \left(\frac{b}{\zeta} + a \right) - \frac{1}{b} \log \left(\frac{b}{\sqrt{E_0}} + a \right),\end{aligned}$$

and solving for ζ yields the lower bound

$$\zeta \geq \frac{b}{a} \left[\left(1 + \frac{b}{a\sqrt{E_0}} \right) e^{b\rho} - 1 \right]^{-1}. \quad (5.23)$$

By (5.21), w is monotone (at least) as long as $\frac{r_0}{8}\zeta - 4 > 0$ (and $0 \leq \rho \leq \bar{\rho}$). By (5.23), this is achieved (at least) for

$$\begin{aligned}\rho &< \frac{1}{b} \log \left(1 + \frac{\frac{1}{32r_0} - \frac{1}{\sqrt{E_0}}}{\frac{a}{b} + \frac{1}{\sqrt{E_0}}} \right) \\ &\rightarrow \frac{1}{b} \log \left(1 + \frac{b}{32r_0a} \right) = \frac{r_0}{r_0 + 16} \log \left(1 + \frac{r_0 + 16}{2^{13} 3r_0} \right),\end{aligned}$$

where the limit is taken as $E_0 \rightarrow \infty$. In particular, we see that w is monotone on $0 \leq \rho \leq \bar{\rho}$ for sufficiently large E_0 , since $\bar{\rho} \leq \delta$, and δ is less than half of the limit above, cf. (5.19). Integrating now (5.23) for the final time, we get

$$\phi(\bar{\rho}) \geq \frac{1}{a} \log \left(1 + \frac{a}{b} (1 - e^{-b\bar{\rho}}) \sqrt{E_0} \right).$$

But $\phi(\bar{\rho})$ is bounded as $E_0 \rightarrow \infty$ by (5.22), so this inequality generates a contradiction unless $\bar{\rho} \rightarrow 0$, which in turn implies that $\bar{\rho} < \delta$ for sufficiently large E_0 and completes the proof. $\square$

Remark. The proof actually shows that the orbit escapes the strip $|w| \leq 1$ arbitrarily fast as $E_0 \rightarrow \infty$, just as one would intuitively expect. However, we do not need this fact so it is not a part of the statement of Theorem 5.5.1.

5.6 Construction of symmetric wormholes

Having performed the analysis of the previous two sections, we finally have all the ingredients to show the main existence result (Theorem 5.0.1) of this manuscript. The theorem will follow by reflection using the symmetries discussed in Remark 5.1.1 and the following result.

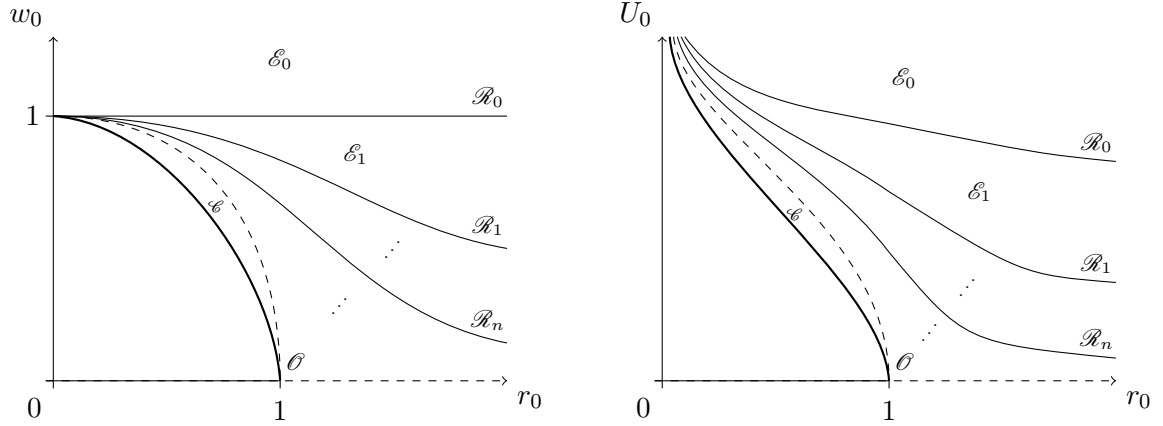


Figure 5.1: A (qualitative) visualization of the set $\mathcal{I}_0$ of admissible initial data with $U_0 = 0$ (left) and $w_0 = 0$ (right). The thick solid lines represent the set where $E_0 = 0$, the dashed lines represent the set of oscillatory orbits $\mathcal{O}$, and the thin solid lines represent the regular orbits $\mathcal{R}_n$.

Theorem 5.6.1. *Fix $r_0 > 0$. For each $n \geq 0$, there exist initial parameters*

$$0 < w_0^{(n)} = w_0^{(n)}(r_0) \leq 1 \quad \text{and} \quad U_0^{(n)} = U_0^{(n)}(r_0) > 0$$

such that a

$$(w_0^{(n)}, 0) \in \mathcal{R}_n(r_0) \quad \text{and} \quad (0, U_0^{(n)}) \in \mathcal{R}_n(r_0).$$

Furthermore, as $n \rightarrow \infty$, we have

$$(w_0^{(n)}, 0) \rightarrow (w_0^{(\infty)}, 0) \in \mathcal{O}(r_0) \quad \text{and} \quad (0, U_0^{(n)}) \rightarrow (0, U_0^{(\infty)}) \in \mathcal{O}(r_0).$$

The existence of the sequence $w_0^{(n)}$ is an analogue of the corresponding result for black hole initial conditions [BFM94, Theorem 34]. The existence of the sequence $U_0^{(n)}$ has no analogue for the black hole setting. The final part of the theorem is not really relevant for the construction of wormholes, but it comes at virtually no additional cost. In any case, it is interesting to know that non-trivial oscillatory orbits exist for $r_0 < 1$, while $w_0^{(n)}, U_0^{(n)} \rightarrow 0$ for $r_0 \geq 1$.

In Figure 5.1, we present a visualization of the set of admissible initial data, where either $U_0 = 0$ or $w_0 = 0$ are kept fixed. The reader can use these figures as a visualization guide throughout the proof.

Proof. We construct the sequences inductively. As in the statement, we will assume that for the even orbits (i.e. those with $U_0 = 0$), we have $w_0 \geq 0$, and for the odd orbits (i.e. those with $w_0 = 0$), we have $U_0 \geq 0$. This is possible due to the symmetry $(w, U) \mapsto -(w, U)$. Note that the boundary condition $E_0 = 0$ of the set $\mathcal{I}_0(r_0)$ of initial

parameters can only be reached if $r_0 \leq 1$. We set

$$w_0^{(\min)} = \begin{cases} \sqrt{1 - r_0}, & \text{if } r_0 < 1, \\ 0, & \text{if } r_0 \geq 1, \end{cases} \quad \text{and} \quad U_0^{(\min)} = \begin{cases} \sqrt{\frac{1}{2}(r_0^{-2} - 1)}, & \text{if } r_0 < 1, \\ 0, & \text{if } r_0 \geq 1, \end{cases}$$

so that $(w_0, 0) \in \mathcal{I}_0(r_0)$ for $w_0^{(\min)} \leq w_0 \leq 1$ and $(0, U_0) \in \mathcal{I}_0(r_0)$ for $U_0^{(\min)} \leq U_0$. Furthermore, by Theorem 5.5.1 (i),

$$(w_0^{(\min)}, 0), (0, U_0^{(\min)}) \in \begin{cases} \mathcal{C}(r_0), & \text{if } r_0 < 1, \\ \mathcal{O}(r_0), & \text{if } r_0 \geq 1. \end{cases}$$

The idea is now to perform a shooting method along the lines

$$\begin{aligned} \mathcal{W}(r_0) &= \mathcal{I}(r_0) \cap \{w_0 \neq 0, U_0 = 0\}, \\ \mathcal{U}(r_0) &= \mathcal{I}(r_0) \cap \{w_0 = 0, U_0 \neq 0\}. \end{aligned}$$

Let¹²

$$\begin{aligned} w_0^{(0)} &= \inf\{w_0 \mid w_0^{(\min)} \leq w_0, (w_0, 0) \in (\mathcal{R}_0 \cup \mathcal{E}_0)(r_0)\}, \\ U_0^{(0)} &= \inf\{U_0 \mid U_0^{(\min)} \leq U_0, (0, U_0) \in (\mathcal{R}_0 \cup \mathcal{E}_0)(r_0)\}. \end{aligned}$$

Note that $w_0^{(0)}$ is well-defined because $(1, 0) \in \mathcal{R}_0(r_0)$ is the trivial solution with $w \equiv 1$, while $U_0^{(0)}$ is well-defined because $(0, U_0) \in \mathcal{E}_0(r_0)$ for sufficiently large U_0 by Theorem 5.5.1 (ii). By Theorem 5.4.2, $(w_0^{(0)}, 0)$ and $(0, U_0^{(0)})$ cannot be in:

- ◇ $\mathcal{E}_n(r_0)$ for $n \geq 0$, or $\mathcal{C}(r_0)$, because these sets are open relative to the lines $\mathcal{W}(r_0)$ and $\mathcal{U}(r_0)$, cf. Theorem 5.4.2 (i),
- ◇ $\mathcal{O}(r_0)$ because this set is neighboured by orbits with arbitrarily many zeros of w , cf. Theorem 5.4.2 (ii),
- ◇ $\mathcal{R}_n(r_0)$ for $n \geq 1$ because each of these sets respectively is neighboured (relative to the lines $\mathcal{W}(r_0)$ and $\mathcal{U}(r_0)$) by orbits with either n or $n + 1$ zeros of w , cf. Theorem 5.4.2 (iii).

Thus, $(w_0^{(0)}, 0)$ and $(0, U_0^{(0)})$ must belong to the last remaining option, namely $\mathcal{R}_0(r_0)$. We also see in particular that $w_0^{(0)} > w_0^{(\min)}$ and $U_0^{(0)} > U_0^{(\min)}$. Furthermore, by construction and Theorem 5.4.2 (iii), the $(w_0, 0)$ - and $(0, U_0)$ -orbits with sufficiently close $w_0 < w_0^{(0)}$ and $U_0 < U_0^{(0)}$ must be in $\mathcal{E}_1$.

¹²With this choice we want to not only show that there are orbits in $\mathcal{R}_0(r_0)$ but we also want to choose the smallest such, as it is not clear whether they are unique. The same will be done when constructing the orbits in $\mathcal{R}_n(r_0)$ for $n \geq 1$. In contrast, in Figure 5.1 we display these sets as if they were unique, because the numerical results suggest that this is indeed the case.

Now for the induction step, suppose $w_0^{(n-1)}, U_0^{(n-1)}$ have been defined, and let

$$\begin{aligned} w_0^{(n)} &= \inf\{w_0 \mid w_0^{(\min)} \leq w_0 \leq w_0^{(n-1)}, (w_0, 0) \in (\mathcal{R}_n \cup \mathcal{E}_n)(r_0)\}, \\ U_0^{(n)} &= \inf\{U_0 \mid U_0^{(\min)} \leq U_0 \leq U_0^{(n-1)}, (0, U_0) \in (\mathcal{R}_n \cup \mathcal{E}_n)(r_0)\}. \end{aligned}$$

Then $(w_0^{(n)}, 0)$ and $(0, U_0^{(n)})$ belong to $\mathcal{R}_n(r_0)$ by an argument analogous to the base case. Furthermore, we also have $w_0^{(n)} > w_0^{(\min)}$ and $U_0^{(n)} > U_0^{(\min)}$, and the $(w_0, 0)$ - and $(0, U_0)$ -orbits with sufficiently close $w_0 < w_0^{(n)}$ and $U_0 < U_0^{(n)}$ must be in $\mathcal{E}_{n+1}$.

Now, $w_0^{(n)}$ and $U_0^{(n)}$ are decreasing bounded sequences, so they converge to some limits $w_0^{(\infty)}$ and $U_0^{(\infty)}$ respectively. The $(w_0^{(\infty)}, 0)$ - and $(0, U_0^{(\infty)})$ -orbits cannot be in any $\mathcal{R}_n(r_0)$ or $\mathcal{E}_n(r_0)$ by construction, nor can they be in $\mathcal{C}(r_0)$ because this set is open relative to the respective lines. Hence, they must be in $\mathcal{O}(r_0)$, completing the proof. $\square$

Proof of Theorem 5.0.1. Fix $r_0 > 0$, let $w_0^{(n)} = w_0^{(n)}(r_0) > 0$ and $U_0^{(n)} = U_0^{(n)}(r_0) > 0$ be the sequences manufactured by Theorem 5.6.1.

The solutions with initial parameters $(r_0, w_0^{(n)}, 0)$ are then well-defined for all $\rho \geq 0$. By the symmetries discussed in Remark 5.1.1, we see that the corresponding functions w, r, ζ are even and U, N, κ are odd. Therefore each of these solutions is also well-defined backwards for all $\rho < 0$ and shares the same properties (but reflected depending on the parity) of its forward counterpart. In particular, for the $(w_0^{(n)}, 0)$ -orbit, w in total has $2n$ zeros (n on each half of the real line), and $w \rightarrow (-1)^n$ as $\rho \rightarrow \pm\infty$.

Similarly, for the solution with initial parameters $(r_0, 0, U_0^{(n)})$, we see that U, r, ζ are even functions while w, N, κ are odd, and the solution is also defined for all $\rho < 0$ with the same (but reflected) properties of its forward counterpart. In particular, for the $(0, U_0^{(n)})$ -orbit, w has $2n + 1$ zeros (n on each half of the real line, and an additional one at $\rho = 0$), and $w \rightarrow \pm(-1)^n$ as $\rho \rightarrow \pm\infty$.

Now, for each $n \geq 0$, set

$$x_0^{(n)} = \begin{cases} (r_0, w_0^{(k)}, 0), & \text{if } n = 2k \text{ is even,} \\ (r_0, 0, U_0^{(k)}), & \text{if } n = 2k + 1 \text{ is odd.} \end{cases}$$

Recall that we have hidden the metric coefficient τ by defining $\zeta = \pi_0/(re^\tau)$ where $\pi_0 = r_0 e^{\tau_0} \sqrt{E_0}$ is the phantom charge, cf. §5.1 and (5.10). We thus define the metric coefficients

$$\begin{aligned} \tau^{(n)}(\rho) &= \log \pi_0 - \log(r\zeta)(\rho, x_0^{(n)}) \\ &= \tau_0 + \log(r_0 \sqrt{E_0}) - \log(r\zeta)(\rho, x_0^{(n)}), \\ r^{(n)}(\rho) &= r(\rho, x_0^{(n)}), \end{aligned}$$

the Yang-Mills potential coefficient

$$w^{(n)}(\rho) = w(\rho, x_0^{(n)}),$$

and the phantom field (4.12)

$$\phi^{(n)}(\rho) = \sqrt{2} \int_0^\rho \zeta(s, x_0^{(n)}) ds.$$

Since τ_0 is still free, we can choose it so as to make $\tau^{(n)} \rightarrow 0$ at $\pm\infty$, since $r\zeta$ tends to some finite limit by Theorem 5.3.1. Then we see that these functions describe symmetric wormhole solutions, as desired in Theorem 5.0.1. Finally, we note that for $r_0 \geq 1$ the variable N has only a single zero located at $\rho = 0$ by Lemma 5.3.9, and this point describes a non-degenerate wormhole throat. On the other hand, when $r_0 < 1$ and n is even (resp. odd), then $\dot{N}_0 > 0$ (resp. $\dot{N}_0 < 0$) by (5.4b), so the initial point $\rho = 0$ describes a non-degenerate wormhole throat (resp. belly). $\square$

Remark 5.6.2. Our results have the obvious weakness of not being able to specify the total number of zeros N has for $r_0 < 1$. One may hope to be able to control the number of zeros of N in the neighbourhood theorem and subsequently the shooting method in a similar way we control the number of zeros of w , but this is unfortunately not possible. The main culprit for this is the fact that nothing prohibits N from having double zeros, i.e. points such that $N = 0$ and $\dot{N} = 0$.¹³ In particular, if an orbit has a double zeros of N , then for the nearby orbits this zero could bifurcate into several zeros, or even completely disappear. Even though we cannot prove this using our current techniques, the numerics indicate that the solutions with $r_0 < 1$ have exactly one non-degenerate throat for even n , and exactly two non-degenerate throats for odd n . Nevertheless, Lemma 5.3.11 ensures at the very least that N has only a finite zeros for the regular solutions, and in particular the wormhole solutions with $r_0 < 1$ have at most finitely many throats/bellies.

As far as asymptotic flatness is concerned, we can show that the wormhole solutions constructed here decay to the Schwarzschild metric with mass m_∞ as $r \rightarrow \infty$ at a suitable rate. By symmetry, we can only focus on the asymptotically flat end $\rho \rightarrow \infty$. Since $N = \dot{r}/r \rightarrow 1$, the function r is monotone for large enough ρ so we may use it as a coordinate and write the metric as

$$g = -e^{2\tau} dt \otimes dt + \frac{dr \otimes dr}{N^2} + r^2 g_{\mathbb{S}^2},$$

where the radial coefficient is obtained via

$$\langle dr, dr \rangle = \dot{r}^2 \langle d\rho, d\rho \rangle = \frac{\dot{r}^2}{r^2} = N^2.$$

We can now state the asymptotic behaviour more precisely. We will use the notation $f \lesssim g$ to mean $f \leq cg$ for some constant $c \geq 0$.

¹³In view of the definitions from §2, such points represent timelike trapping horizons with vanishing surface gravity.

Proposition 5.6.3. *If (τ, r, w) is a symmetric haunted hairy $\mathbf{SU}(2)$ wormhole solution from Theorem 5.0.1 with mass m_∞ , then*

$$\left| e^{2\tau} - \left(1 - \frac{2m_\infty}{r} \right) \right| \lesssim \frac{1}{r^2},$$

$$\left| N^2 - \left(1 - \frac{2m_\infty}{r} \right) \right| \lesssim \frac{1}{r^2},$$

for sufficiently large r .

Remark. The reader should compare this result with [SWY93, §4], where such a result is obtained for black hole solutions in pure Einstein-Yang-Mills theory. In fact, such a result can be obtained for any solution belonging to class (iii) of Theorem 5.3.1, although one should then be careful with how τ is scaled at infinity (which is also why we deferred this result up until here).

Proof. First, we note that the mass satisfies (cf. (2.5, 5.14))

$$|m - m_\infty| \leq \int_\rho^\infty |\dot{m}| = \frac{1}{2} \int_\rho^\infty \frac{\dot{r}}{r^2} |2\dot{w}^2 + (1 - w^2)^2 - (r\zeta)^2| \lesssim \int_\rho^\infty \frac{\dot{r}}{r^2} = \frac{1}{r},$$

for large ρ , where we use the facts that $(w, \dot{w}) \rightarrow (1, 0)$, $N \rightarrow 1$ and $r\zeta \rightarrow \alpha \in \mathbb{R}$ (cf. Lemma 5.3.13). Then it follows that

$$\left| N^2 - \left(1 - \frac{2m_\infty}{r} \right) \right| = \frac{2|m - m_\infty|}{r} \lesssim \frac{1}{r^2}.$$

For the temporal coefficient, we put $e^\tau = Ne^\delta$ and note that

$$\left| e^{2\tau} - \left(1 - \frac{2m_\infty}{r} \right) \right| \leq N^2 |e^{2\delta} - 1| + \left| N^2 - \left(1 - \frac{2m_\infty}{r} \right) \right|,$$

so the claim will follow if the first term on the right hand side is bounded by a multiple of $1/r^2$ for large r . To this end, note that $\delta \rightarrow 0$ since $e^\delta = \frac{e^\tau}{N} \rightarrow 1$, and using (5.6), we can estimate

$$\begin{aligned} |e^{2\delta} - 1| &\leq \int_\rho^\infty \left| \frac{d}{d\rho} \left(\frac{e^{2\tau}}{N^2} \right) \right| = 2 \int_\rho^\infty \frac{e^{2\delta}}{N^3} |2U^2 - \zeta^2| \\ &\lesssim \int_\rho^\infty \frac{1}{r^2} |2\dot{w}^2 - (r\zeta)^2| \lesssim \int_\rho^\infty \frac{1}{r^2} \lesssim \int_\rho^\infty \frac{\dot{r}}{r^3} = \frac{2}{r^2}, \end{aligned}$$

where we repeatedly use the fact that $N \rightarrow 1$. □

n	m	w_0	U_0
0	2	0.065749965122	2.387813074897
0	3	0.084835112960	2.275028141102
0	4	0.088404573457	2.254612018293
0	5	0.088999225383	2.251230932725
1	3	0.018829723491	1.961260476747
1	4	0.022343374339	1.945208275470
1	5	0.022928486454	1.942546748413
2	4	0.003511646294	1.864011058565
2	5	0.004096369202	1.861520432080
3	5	0.000584712822	1.846718238329

Table 5.2: Initial data describing asymmetric wormholes for $r_0 = 0.75$.

5.7 Asymmetric wormholes

In [Hau+14, §4], it was proposed that it might be interesting to study also asymmetric solutions. In fact, our numerical analysis suggests that there exist such wormhole solutions, having n zeros of w for $\rho > 0$ and m zeros for $\rho \leq 0$. In Table 5.2, we list our numerical findings of initial data describing such solutions. We also plot the corresponding solutions for certain choices of n and m in Figure 5.5 (at the end of the chapter).

In view of this, we conjecture the following asymmetric extension of Theorem 5.0.1.

Conjecture 5.7.1. *For each $0 < r_0 < 1$ and $0 \leq n < m$, there exists an asymmetric haunted hairy $\text{SU}(2)$ wormhole*

$$(\tau^{(n,m)}, r^{(n,m)}, w^{(n,m)})$$

such that $w^{(n,m)}$ has n zeros for $\rho > 0$ and m zeros for $\rho \leq 0$.

Remark. Note that the even (resp. odd) solutions obtained in Theorem 5.0.1 would in this notation have $n = m$ (resp. $m = n + 1$), which is why our numerical results only display asymmetric solutions with $m - n \geq 2$.

Let us present also some analytical details that could be used to prove Conjecture 5.7.1. First, we note that Theorem 5.6.1 can in fact easily be extended.

Theorem 5.7.2. *Fix $0 < r_0 < 1$. For each w_0 with $r_0 + w_0^2 \leq 1$, there exist initial parameters $U_0^{(n)}(w_0) = U_0^{(n)}(r_0, w_0) > 0$ such that*

$$(w_0, U_0^{(n)}(w_0)) \in \mathcal{R}_n(r_0), \quad \text{and} \quad (w_0, U_0^{(n)}(w_0)) \rightarrow (w_0, U_0^{(\infty)}(w_0)) \in \mathcal{O}(r_0),$$

for each $n \geq 0$ (resp. $n \geq 1$) if $w_0 \geq 0$ (resp. $w_0 < 0$).

Proof. Setting

$$U_0^{(\min)}(w_0) = \sqrt{\frac{1}{2} \left(\frac{(1 - w_0^2)^2}{r_0^2} - 1 \right)},$$

we see that $(w_0, U_0^{(\min)}(w_0)) \in \mathcal{C}(r_0)$ by Theorem 5.5.1 (i), since $E_0 = 0$ in this case. On the other hand, $(w_0, U_0) \in (\mathcal{E}_0 \cup \mathcal{E}_1)(r_0)$ and w is monotone for sufficiently large $U_0 > 0$ by Theorem 5.5.1 (ii), so we can apply the same argument as in the proof of Theorem 5.6.1 to obtain the desired sequence for each fixed r_0 and w_0 , as well as its limit. $\square$

Remark. The condition $r_0 + w_0^2 \leq 1$ is necessary because we would like to have a U_0 for which $(w_0, U_0) \in \mathcal{C}(r_0)$ in order to perform the shooting method. In the case $r_0 > 1$ of Theorem 5.6.1, this was not needed because our set of eligible values of U_0 contained the trivial solution $w \equiv 0$ belonging to $\mathcal{O}(r_0)$, which also serves well enough as a lower bound for the shooting method.

We thus obtain a sequence of solutions which are defined for all $\rho \geq 0$ with exactly n zeros of w , and which have the correct boundary behaviour at $\rho = \infty$, with neither w_0 nor U_0 being zero. However, these might not be defined for all $\rho < 0$ nor do they need to have incorrect behaviour at $\rho = -\infty$ (i.e. they might be asymptotically cylindrical rather than flat), since they are no longer symmetric. Nevertheless, the identities from Remark 5.1.1 imply that the backwards solution will have the desired behaviour if

$$U_0^{(n)}(w_0) = U_0^{(m)}(-w_0), \text{ for some } -\sqrt{1 - r_0} \leq w_0 \leq \sqrt{1 - r_0} \text{ and } n < m,$$

so the goal is to find w_0 having this property. Due to the way we count zeros of w (cf. Definition 5.4.1), it however turns out that the functions $U_0^{(n)}(w_0)$ will have a discontinuity at $w_0 = 0$ for each n , so it makes more sense in this context to define

$$\tilde{U}_0^{(n)}(w_0) = \begin{cases} U_0^{(n)}(w_0), & \text{if } w_0 \geq 0, \\ U_0^{(n+1)}(w_0), & \text{if } w_0 < 0. \end{cases}$$

Note that, by construction, $\tilde{U}_0^{(n)}(w_0)$ decreases with n for fixed w_0 , since $U_0^{(n)}(w_0)$ does. The numerics suggest even more.

Conjecture 5.7.3. *For each $n \geq 0$, the function $w_0 \mapsto \tilde{U}_0^{(n)}(w_0)$ is continuous, decreasing, and*

$$\tilde{U}_0^{(\infty)}(-\sqrt{1 - r_0}) \geq \tilde{U}_0^{(0)}(\sqrt{1 - r_0}).$$

If one could show this conjecture, then the desired w_0 could be extracted by applying the intermediate value theorem to the functions

$$\Phi_{m,n} : w_0 \mapsto \tilde{U}_0^{(n)}(w_0) - \tilde{U}_0^{(m-1)}(w_0), \quad n < m,$$

because it has opposite signs at each endpoint of the w_0 -interval. We visualize this behaviour in Figure 5.2, where each intersection between a solid and a dashed curve

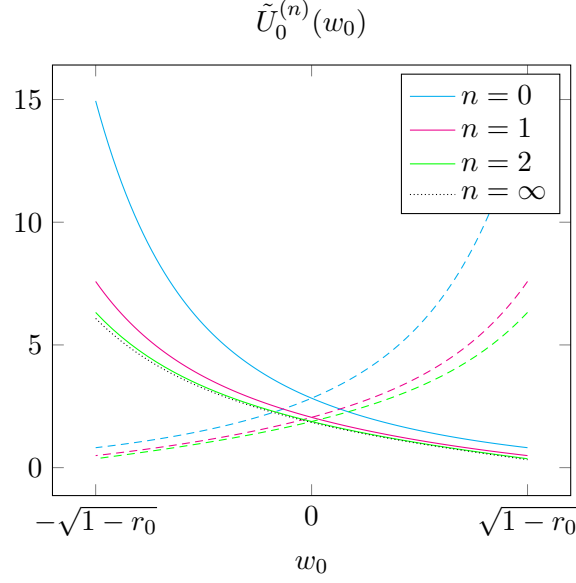


Figure 5.2: The functions $\tilde{U}_0^{(n)}(w_0)$ (solid), their reflections $\tilde{U}_0^{(n)}(-w_0)$ (dashed), and the limit $\tilde{U}_0^{(\infty)}(w_0)$ (dotted), for $r_0 = 0.75$. Every intersection between a solid and a dashed curve represents initial data describing a wormhole. The intersections with $w_0 = 0$ are precisely the odd solutions from Theorem 5.0.1.

represents a zero of some $\Phi_{n,m}$, and hence a wormhole with n zeros of w for $\rho > 0$ and m zeros for $\rho \leq 0$. The values given in Table 5.2 are precisely these intersections.

Be that as it may, there unfortunately does not seem to be a proof of Conjecture 5.7.3 in sight. Indeed, we face here similar difficulties as when trying to prove uniqueness of regular orbits with a given number of zeros (with r_0 and either w_0 or U_0 fixed), cf. discussion below Theorem 5.4.2 and [BFM94, end of §8]. We therefore leave the statement merely as a conjecture.

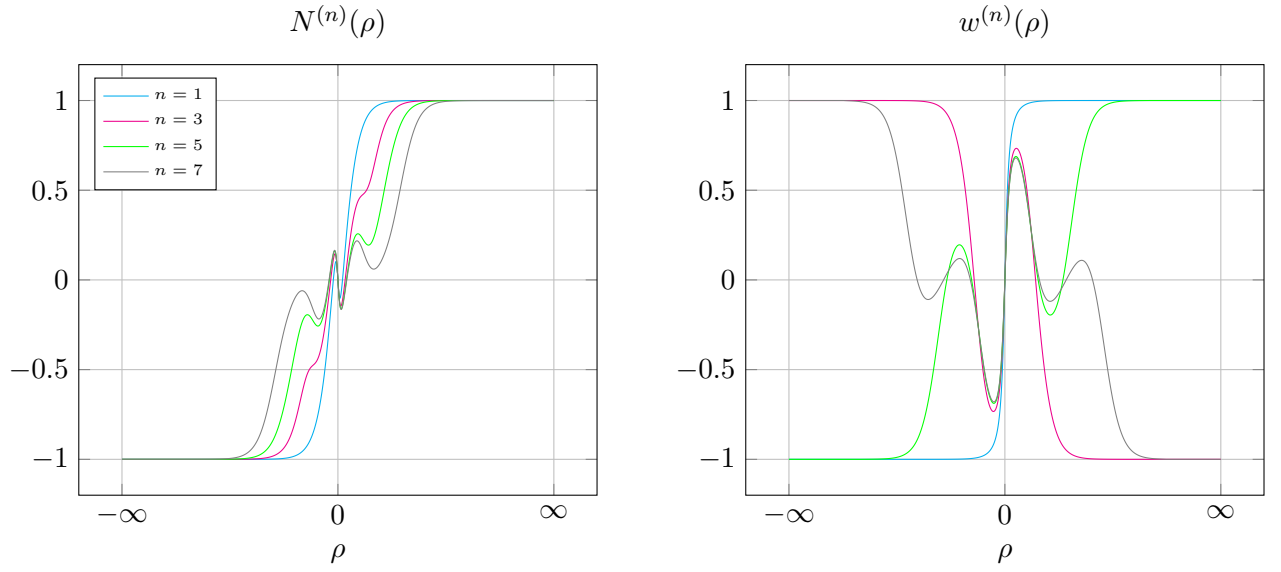


Figure 5.3: Odd symmetric wormhole solutions for $r_0 = 0.75$. The plots of the other dependent variables are not particularly inspiring so they are omitted.

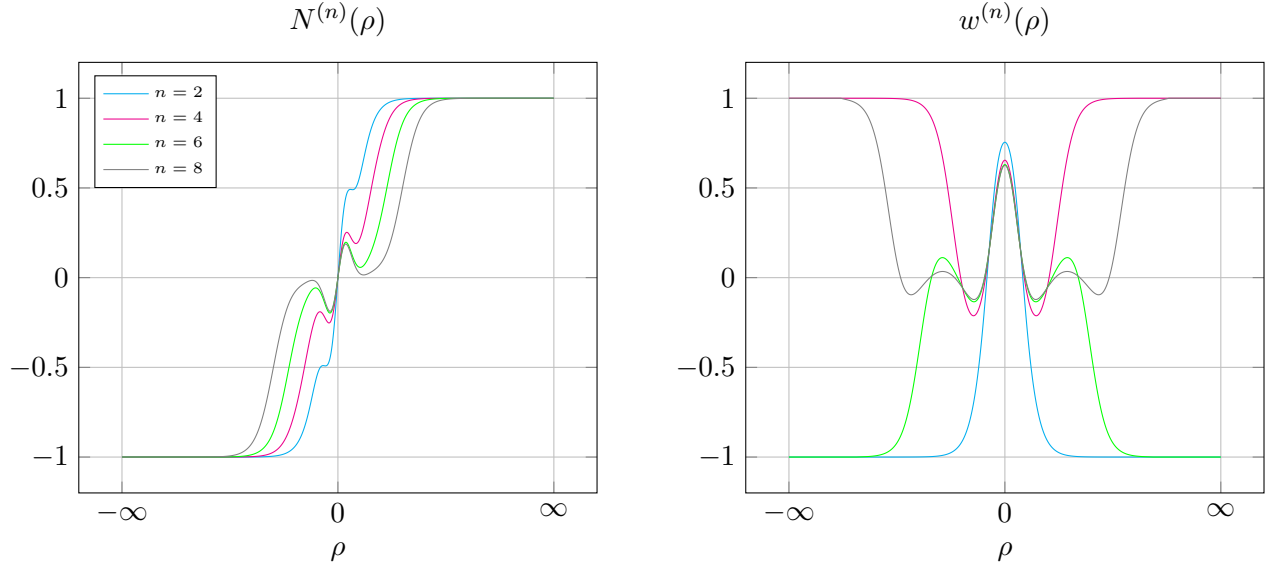


Figure 5.4: Even symmetric wormhole solutions for $r_0 = 0.75$. The plots of the other dependent variables are not particularly inspiring so they are omitted.

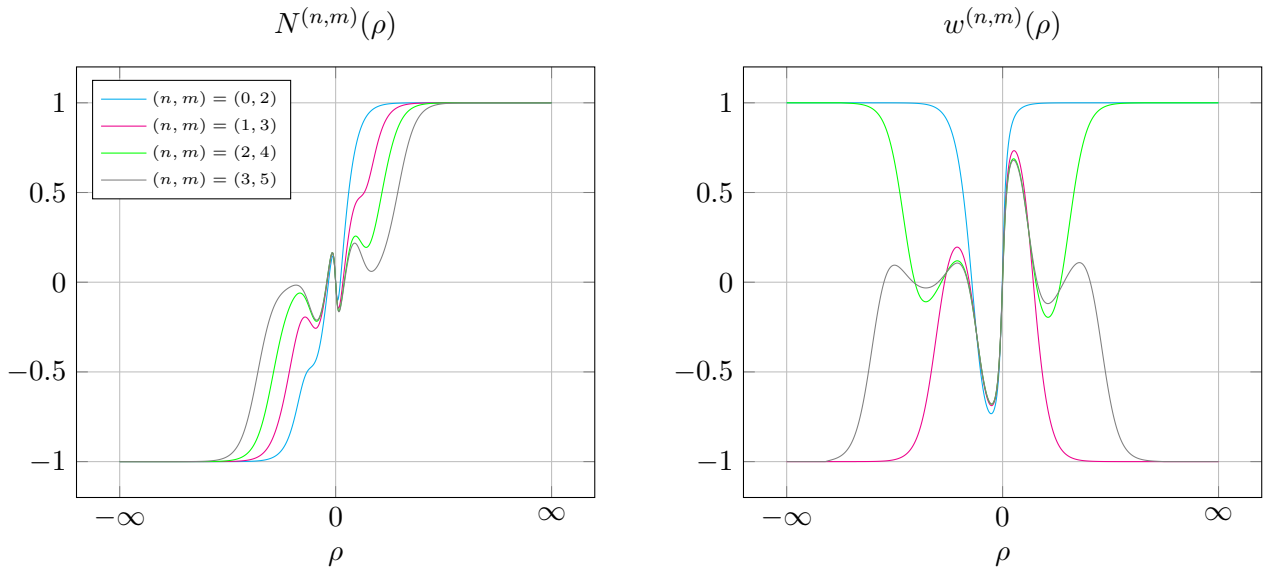


Figure 5.5: Asymmetric wormhole solutions for $r_0 = 0.75$. Plotted using the initial data given in Table 5.2. The plots of the other dependent variables are not particularly inspiring so they are omitted.

Chapter 6

Outlook

Having wrapped up §5, in this final chapter we would like to discuss some problems to the haunted $\mathbf{SU}(2)$ EYM theory that may be interesting to study in the future, as well as some generalizations of the theory that we have considered.

6.1 Related problems

The structure of the $\mathbf{SU}(2)$ EYM equations and its solutions interestingly appears in many other geometric problems. Here, by structure we mean the existence of infinitely many global solutions to an ODE system, labelled by the number of zeros of a solution (sometimes called the *nodal number*). For example, one can use a similar shooting method technique as we did in this work to construct infinite sequences of harmonic (and more generally p -harmonic) maps between spheres [BC97; BS23]. In many problems where such techniques are employed, the numerical analysis heavily suggests that the global solutions furnished by the shooting method are in fact also *uniquely* determined by their nodal number (up to possible symmetries). In the context of our main theorem (Theorem 5.0.1), this means that we would like to not only say that the sequence of symmetric wormhole solutions exists, but also that it is the unique such sequence of symmetric wormhole solutions (for fixed r_0) modulo the symmetry $w \mapsto -w$. Stated more visually, one would like to show that the set of initial data looks as in Figure 5.1, where the regular initial data are described by curves which are injective as functions of w_0 (resp. U_0). However, such uniqueness results do not seem to be known even in the simpler cases of harmonic maps mentioned above. Some attempts in this direction have been made for the pure $\mathbf{SU}(2)$ EYM equations in [BFM94] (cf. also Remark 5.4.3), where the authors reduce the problem of uniqueness to showing that a certain function is monotone with respect to the initial data. This reduction is in some sense similar to our Conjecture 5.7.3, which also reduces the problem of existence for asymmetric wormhole solutions to a monotonicity property of a certain function with respect to initial data. However, actually showing the monotonicity of such functions does not seem feasible at the moment, and would likely require the development of completely new techniques. Nevertheless, it would be interesting to attack this problem in the future, as its resolution

could potentially be widely applicable.

Going back to existence results, we would also like to recall that we have assumed the initial value $\kappa(0) = 0$ throughout the manuscript. This is, however, not a necessary condition (cf. §4.3), and non-zero values of $\kappa(0)$ should likely also lead to sequences of wormhole solutions. Note that such wormholes would necessarily be asymmetric, so that one should probably first have a good understanding of how these can be constructed for the simpler case $\kappa(0) = 0$, as discussed in §5.7.

Another directly related problem, and in particular a weakness of the techniques used in this manuscript, is that we are not able to count the number of trapping horizons that the constructed wormholes have in the case $r_0 < 1$ in a satisfactory way, nor are we really able to say whether the trapping horizons are degenerate or not (in the sense of §2.2–2.3), except for the ones located at the initial point $\rho = 0$. We have already discussed this in some length in Remark 5.6.2. Since trapping horizons play such an important role in wormhole theory, it would be interesting to revisit this problem again in the future.

Finally, there is also the natural question of the stability of the wormholes constructed in this work. It is well-known that the Ellis-Bronnikov family of wormholes is unstable both under linear spherically symmetric perturbations [GGS08a] and in the non-linear sector [GGS08b]. Attempts have been made to stabilize these wormholes by introducing electric charge (thus obtaining the abelian wormhole family from Table 4.1), but such wormholes turned out to be unstable as well [GGS09]. In fact, it is also known that the hairy black hole solutions in pure Einstein-Yang-Mills theory are also unstable [VG99; BS96]. Therefore the stability of the (non-trivial) haunted hairy $\mathbf{SU}(2)$ wormholes would come as somewhat of a surprise if it were true, and it seems more likely that one would have to look at weaker notions of stability to get some positive results. In either case, it would be interesting to investigate this properly and see what can be shown.

6.2 Non-minimal coupling

Throughout this manuscript we have considered the Einstein-Yang-Mills theory with a so-called minimal coupling to a phantom field, due to the lack of interaction between the Yang-Mills and the phantom Lagrangian. In this section we would like to point out a few initial remarks for studying also non-minimal couplings, but we leave a more extensive analysis is left for a future project.

The non-minimal haunted Einstein-Yang-Mills theory is a local Lagrangian theory with respect to

$$L_f : (g, \omega, \phi) \mapsto (R_g - f(\phi)\|F_\omega\|^2 + \|d\phi\|^2) \operatorname{vol}_g$$

where $f \in C^1(\mathbb{R})$ is a given fixed positive function, cf. §4.1. Two common choices of f are [Ast+19]:

- ◊ the *dilatonic* coupling $f(\phi) = e^{\lambda\phi}$ for a constant $\lambda \in \mathbb{R}$ (such couplings often appear in supergravity/low-energy string theory models, and also pop up quite naturally in the context of Kaluza-Klein theory);

◇ the coupling $f(\phi) = e^{\lambda\phi^2}$ for a constant $\lambda \in \mathbb{R}$ (such couplings appear in certain cosmological models [MSS13]).

A simple modification of Theorem 4.1.2 shows that the corresponding Euler-Lagrange equations are

$$\left\{ \begin{array}{l} \text{Ric}_g - 2f(\phi)\langle F_\omega \otimes F_\omega \rangle + \frac{1}{n-2}f(\phi)\|F_\omega\|^2g + d\phi \otimes d\phi = 0, \\ d_\omega \star (f(\phi)F_\omega) = 0, \\ \square_g \phi + \frac{1}{2}f'(\phi)\|F_\omega\|^2 = 0. \end{array} \right.$$

We set $f = e^h$ for some function h and employ the same ansätze for the metric, connection, and phantom field, as in Theorem 4.2.1. The corresponding equation system becomes

$$\left\{ \begin{array}{l} \ddot{w} + \left(\dot{\tau} - \frac{\dot{r}}{r} + h'(\phi)\dot{\phi} \right) \dot{w} + w(1 - w^2) = 0, \\ \ddot{\phi} + \left(\dot{\tau} + \frac{\dot{r}}{r} \right) \dot{\phi} + h'(\phi) \frac{e^{h(\phi)}}{r^2} (2\dot{w}^2 + (1 - w^2)^2) = 0, \\ \ddot{\tau} + \dot{\tau}^2 + \frac{\dot{r}\dot{\tau}}{r} - \frac{e^{h(\phi)}}{r^2} (2\dot{w}^2 + (1 - w^2)^2) = 0, \\ \frac{\ddot{r}}{r} + \frac{\dot{r}\dot{\tau}}{r} - 1 + \frac{e^{h(\phi)}}{r^2} (1 - w^2)^2 = 0, \\ 1 + \frac{e^{h(\phi)}}{r^2} (2\dot{w}^2 - (1 - w^2)^2) - \frac{\dot{r}}{r} \left(2\dot{\tau} + \frac{\dot{r}}{r} \right) - \frac{\dot{\phi}^2}{2} = 0. \end{array} \right.$$

Trying to keep as much of the same form of the equations as for the minimal case, we now define the variables

$$R = re^{-\frac{h(\phi)}{2}}, \quad N = \frac{\dot{r}}{r}, \quad V = \frac{\dot{w}}{R}, \quad \kappa = \dot{\tau} + N, \quad \zeta = \frac{\dot{\phi}}{\sqrt{2}}.$$

In terms of these variables, the system becomes

$$\dot{R} = R \left(N - \frac{1}{2} h'(\phi) \zeta \right), \tag{6.3a}$$

$$\dot{N} = 1 - \frac{(1 - w^2)^2}{R^2} - \kappa N, \tag{6.3b}$$

$$\dot{w} = rU, \tag{6.3c}$$

$$\dot{V} = -(\kappa - N)V - \frac{w(1 - w^2)}{R}, \tag{6.3d}$$

$$\dot{\kappa} = 1 + 2V^2 - \kappa^2, \tag{6.3e}$$

$$\dot{\zeta} = -\kappa\zeta - h'(\phi) \left(2V^2 + \frac{(1 - w^2)^2}{R^2} \right), \tag{6.3f}$$

$$\dot{\phi} = \zeta, \tag{6.3g}$$

together with the constraint

$$\zeta^2 = 1 + N^2 + 2V^2 - \frac{(1 - w^2)^2}{R^2} - 2\kappa N.$$

In particular, we see that the only changes appear in the equations for the generalized radial function R (6.3a) and the phantom term ζ (6.3f). In fact, the theory only depends on the derivative of $h = \log f$, rather than h itself.

If we wish to preserve the symmetries as in Remark 5.1.1, we should require that κ is odd and ζ is even when either $w_0 = 0$ or $U_0 = 0$. In particular, we see from (6.3f–6.3g) that ϕ should be odd (up to a possible translation) and so should h' . We should therefore require that h (and thus also f) is even. In particular, this already prohibits the dilatonic coupling $f(\phi) = e^{\lambda\phi}$ (but e.g. the cosmological coupling $f(\phi) = e^{\lambda\phi^2}$ is permitted).

Now one would ideally like to follow the blueprint of the proof in §5 to prove the existence of sequences of wormhole solutions with non-minimal coupling. The main difficulty here is that the phantom equation (6.3f) is now more heavily coupled to the rest of the system. In contrast, for non-minimal coupling we could immediately conclude certain properties of ζ , which was paramount for the rest of the proof. In the present case, one should probably impose further constraints on the function h to make the problem more feasible. For example, if one assumes that h is non-decreasing, then ζ would still be a decreasing function by (6.3f). However, in this case the region $\zeta > 0$ is no longer invariant and ζ could reach negative values. On the other hand, one could also assume that h is non-increasing, in which case the region $\zeta > 0$ would still be invariant, but in that case ζ would no longer necessarily be decreasing. At the moment, it is not clear which assumptions are appropriate here, but it would be interesting to look into this problem in more detail, starting from a numerical analysis for some specific choices of f .

6.3 Further generalizations

Throughout this manuscript, we have mainly studied the haunted Einstein-Yang-Mills theory with gauge group $G = \mathbf{SU}(2)$. In fact, one can use the general theory of invariant connections over principal G -bundles with symmetry (cf. §3.2) to derive the EYM equations for any gauge group G [BS93]. The resulting equations are of course rather difficult in their full generality and, to the author's best knowledge, only some very partial non-existence results are known in such cases [BS93; BS96]. Nevertheless, one could also derive the haunted EYM equations in such a more general setting, and see if this sheds some new light on the problem. Aside from this, the general theory from §3.2 could also be used to study spacetimes of dimension other than four, and also admitting different symmetry groups.

Finally, we would like to note that there seem to be many other research works where wormholes (with or without angular momentum) supported by different matter fields are constructed numerically without proof. Particularly interesting are some quantum field theory matter fields such as Yang-Mills-Higgs fields [CL20] or Dirac-Maxwell fields [BKR21], just to name a few. It would be interesting to see whether the findings of such works could also be rigorized.

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