

p -Harmonic Functions on Riemannian Lie Groups

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Abstract

This Master's thesis is a study of complex-valued p -harmonic functions on Riemannian manifolds i.e. functions $\phi : (M, g) \rightarrow \mathbb{C}$ which satisfy the $2p$ -th order partial differential equation

$$\tau^p(\phi) = \underbrace{\tau \cdots \tau}_{p \text{ times}}(\phi) = 0,$$

where τ denotes the Laplace-Beltrami operator on the Riemannian manifold (M, g) . Since every p -harmonic function is a fortiori r -harmonic for any $r \geq p$, one is usually interested in proper p -harmonic functions i.e. p -harmonic functions which are not $(p - 1)$ -harmonic.

In this thesis we develop a general method of constructing proper p -harmonic functions of the form $f \circ \phi$, where $f : \phi(M) \subset \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and $\phi : (M, g) \rightarrow \mathbb{C}$ is a so-called isoparametric function on the Riemannian manifold (M, g) . We apply this method to the special case when the isoparametric function ϕ is an eigenfunction on M , and we present examples of eigenfunctions on classical semisimple Lie groups. Finally, we employ the developed method to construct proper p -harmonic functions on the solvable semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$, where $\mathbb{H}^{2n+1}$ denotes the $(2n + 1)$ -dimensional Heisenberg group. The study of these particular Lie groups is motivated by the fact that all four-dimensional Lie groups whose Lie algebra is indecomposable are, up to isomorphism, semidirect products of one of these two types.

Keywords: analysis on Lie groups, Laplace-Beltrami operator, p -harmonic functions, semisimple Lie groups, low-dimensional solvable Lie groups

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well-known for a reference to be given.

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Chapter 1

Introduction

The biharmonic equation $\Delta^2\phi = \Delta\Delta\phi = 0$ has been known and studied for more than two centuries. The first to consider it were the French mathematicians Lagrange and Germain, who derived this equation while studying vibration of thin elastic plates at the beginning of the 1800s. Later in the century, Maxwell discovered that Airy's stress functions, which are central objects in two-dimensional elasticity theory, are biharmonic. Aside from elasticity theory, the biharmonic equation also makes an appearance in two-dimensional hydrodynamics problems involving Stokes flows of incompressible Newtonian fluids. A more comprehensive review of this fascinating history of biharmonic functions can be found in the article [33].

At the end of the 19th century, mathematicians started considering a natural generalization of the biharmonic equation, namely the p -harmonic equation

$$\Delta^p\phi = \underbrace{\Delta \cdots \Delta}_{p \text{ times}}\phi = 0, \quad p \geq 1,$$

the solutions of which are called *p-harmonic functions*. Already in 1899, Almansi [1] observed that any p -harmonic function on $\mathbb{R}^n$ can be represented in terms of harmonic functions. The modern version of his result, which today is known as Almansi's theorem, states that if $\phi_1, \dots, \phi_p$ are harmonic functions on an open set $U \subset \mathbb{R}^n$, then the function ϕ defined by

$$\phi(x) = \sum_{k=1}^p |x|^{2k-2} \phi_k(x)$$

is p -harmonic on U , and conversely that any p -harmonic function on U is of this form, given that U is a star-shaped domain centered at the origin, see [3]. The general theory of the p -harmonic equation and the corresponding boundary value problems on the Euclidean space $\mathbb{R}^n$ is well-understood today. For an extensive analysis of this subject, we refer to the standard texts [3, 15].

There is a natural way of extending the definition the Laplace operator Δ to the more general setting when the underlying space is not the flat Euclidean space, but any Riemannian manifold (M, g) . The linear differential operator then obtained is usually called the *Laplace-Beltrami operator* (or *tension field*) and will here be denoted by τ . Motivated by the Euclidean case, it seems both natural and important to study the p -harmonic equation $\tau^p(\phi) = 0$ on Riemannian manifolds. However,

even though the theory on p -harmonic functions on $\mathbb{R}^n$ is well-known, this is not the case when it comes to p -harmonic functions on general Riemannian manifolds, where the theory appears to be considerably more difficult and there exist much fewer publications.

In [11, 12, 13], Caddeo et al. investigate the biharmonicity of the function r^k , where r denotes the distance function on the underlying Riemannian manifold, and Schimming extends this study in [40] by also considering the p -harmonicity of the function r^k . Their interest in studying the function r^k is motivated by the fact that Almansi's theorem implies, in particular, that the function r^{2p-2} is p -harmonic on $\mathbb{R}^n$, where $r(x) = |x|$ is the distance function on $\mathbb{R}^n$. In the papers it is proven among other things that r^k can be biharmonic on an n -dimensional Riemannian manifold (M, g) only if $k = 2, 2 - n$, or $4 - n$. For some of these cases, further classifications of the underlying manifold are also obtained. For example, if r^{2-n} is biharmonic, then the underlying Riemannian manifold must either be locally flat or a three-dimensional manifold of constant curvature.

In the paper [34], Montaldo and Ratto make an attempt to generalize Almansi's theorem to a certain class of warped products. Namely, they study manifolds of the form $M = [0, \infty) \times \mathbb{S}^n$ equipped with the Riemannian metric

$$g = dr \otimes dr + f(r)^2 g_{\mathbb{S}^n},$$

where $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is the standard unit n -sphere and f is a sufficiently nice function. Motivated by Almansi's theorem, they study the existence of a radial function $\psi = \psi(r)$ with the property that $\psi\phi$ is biharmonic for any harmonic function ϕ . They prove that the only manifold M possessing such a function ψ is flat $\mathbb{R}^{n+1}$ i.e. that in this case we must have $f(r) = r$. In hopes of finding a positive result, they also consider manifolds of the form $M = \mathbb{S}^n \times [0, \infty) \times \mathbb{S}^m$ equipped with the metric

$$g = f_1(r)^2 g_{\mathbb{S}^m} + dr \otimes dr + f_2(r)^2 g_{\mathbb{S}^n}.$$

In this case, they show that $r\phi(r)$ is biharmonic whenever ϕ is a radial harmonic function, and they also provide an example which shows that the assumption that ϕ is radial is essential.

A different approach is taken in [6] by Baird, Fardoun and Ouakkas, who study functions $\phi : (M, g) \rightarrow \mathbb{R}$ which are *isoparametric* i.e. such that

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad \kappa(\phi, \phi) = \Psi \circ \phi$$

for some smooth functions Φ and Ψ , where κ is the bilinear *conformality operator* defined by

$$\kappa(\phi, \psi) = g(\text{grad } \phi, \text{grad } \psi).$$

Rather than working with a fixed metric g , they investigate how the metric g must be deformed in order for an isoparametric function to be biharmonic.

During the recent years, research in this field has been geared towards the construction of examples of p -harmonic functions on classical Riemannian manifolds. In the article [21], Gudmundsson, Montaldo and Ratto construct the first known examples of biharmonic functions on the classical compact simple Lie groups

$$\mathbf{SO}(n), \quad \mathbf{SU}(n), \quad \mathbf{Sp}(n).$$

Gudmundsson and Siffert generalize this construction method in their paper [25]. Biharmonic functions on the celebrated three-dimensional Thurston geometries

$$\mathbf{Sol}, \quad \mathbf{Nil}, \quad \widetilde{\mathbf{SL}}_2(\mathbb{R}), \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}$$

are constructed by Gudmundsson in [19], and a generalization of these results can be found in the recent paper [26] by Gudmundsson and Siffert.

In [27], Gudmundsson and the author of this thesis devise a general method of constructing p -harmonic functions on Riemannian manifolds. This method relies on the existence of *eigenfunctions* on the Riemannian manifold (M, g) , which are defined as smooth functions $\phi : (M, g) \rightarrow \mathbb{C}$ such that the tension field and the conformality operator satisfy

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu\phi^2$$

for some constants $\lambda, \mu \in \mathbb{C}$. In the same paper, the authors present examples of eigenfunctions on the classical semisimple Lie groups

$$\mathbf{SL}_n(\mathbb{R}), \quad \mathbf{Sp}(n, \mathbb{R}), \quad \mathbf{SO}(p, q), \quad \mathbf{SU}(p, q), \quad \mathbf{Sp}(p, q), \quad \mathbf{SO}^*(2n), \quad \mathbf{SU}^*(2n).$$

This also implies the existence of p -harmonic functions on these Lie groups following the main result of the paper.

This thesis follows the same research theme, in that our goal is to construct p -harmonic functions on Riemannian manifolds, with particular focus on certain Riemannian Lie groups. To achieve this goal, we devise a general method of construction which uses complex-valued isoparametric functions. This method generalizes the main result of the paper [27]. To demonstrate how the method can be applied, we consider the semidirect products of the form $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$, where $\mathbb{H}^{2n+1}$ denotes the $(2n+1)$ -dimensional Heisenberg group. Our interest in studying such semidirect products comes from the fact that, according to the classifications in [37, 9], all simply connected solvable three- and four-dimensional Lie groups whose Lie algebra is indecomposable are semidirect products of one of the following types:

$$\mathbb{R} \ltimes \mathbb{R}^2, \quad \mathbb{R} \ltimes \mathbb{R}^3, \quad \mathbb{R}^2 \ltimes \mathbb{R}^2, \quad \mathbb{R} \ltimes \mathbb{H}^3.$$

The text is organised as follows.

Chapter 2: We recall some notions and results from Lie theory and Riemannian geometry which are required for the understanding of this work.

Chapter 3: We define p -harmonic functions and develop a general method of constructing p -harmonic functions using isoparametric functions, which generalizes the method introduced in [27].

Chapter 4: We provide a detailed exposition of the paper [27]. In particular, we apply the above mentioned method to the special case when the isoparametric function is an eigenfunction and we construct eigenfunctions on the classical semisimple Lie groups.

Chapter 5: We study the Lie groups $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$. We supply a linear representation of such Lie groups and we calculate the canonical left-invariant metric in their respective standard linear representations. Finally, we construct p -harmonic functions on these Lie groups, mainly using the theory developed in the previous chapters.

Chapter 2

Preliminaries

We assume that the reader is familiar with some basic theory of smooth Riemannian manifolds, as well as some basic Lie theory. In what follows we introduce the notations that are used throughout the text and we recall some results from these subjects which are essential for the understanding of this work. For a detailed review of Riemannian geometry and Lie theory we recommend the excellent texts [22, 30] and [31, 28], respectively.

2.1 General notation

For a matrix $A \in \mathbb{C}^{n \times n}$ we denote by A_{ij} the (i, j) -th component of A , and by A^{ij} the (i, j) -th component of its inverse A^{-1} given that the inverse exists. We denote the transpose of A by A^T . The trace of the matrix A will be denoted by $\text{Tr}(A)$. For two vectors $v, w \in \mathbb{C}^n$, we denote their complex-bilinear scalar product by $\langle v, w \rangle = \sum_i v_i w_i$. A vector $v \in \mathbb{C}^n$ is said to be *isotropic* if $\langle v, v \rangle = 0$.

If $f : U \rightarrow \mathbb{C}$ is a holomorphic function defined on a simply connected domain $U \subset \mathbb{C}$ we will use the notation

$$z \mapsto \int^z f(\zeta) d\zeta$$

to denote any particular choice of antiderivative of the function f . The same notation will be used if $f : U \subset \mathbb{R} \rightarrow \mathbb{C}$ is continuous.

Given a smooth manifold M , we denote by $T_p M$ and $T_p^* M$ its tangent and cotangent spaces at $p \in M$ and by TM and T^*M its tangent and cotangent bundles, respectively. If $(x_1, \dots, x_n)$ is a local coordinate system, the induced *local frame* of TM and *local coframe* of T^*M will be denoted by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ and $dx_1, \dots, dx_n$, respectively. The space of r -times continuously differentiable maps $f : M \rightarrow N$ between two smooth manifolds M and N will be denoted by $C^r(M, N)$. The space of r -times continuously differentiable sections of a smooth vector bundle E over M will be denoted by $C^r(E)$. Sections of the tangent bundle TM are called *vector fields*. For most purposes of this thesis, all functions and bundle sections can be assumed to be smooth i.e. of the class C^∞ .

2.2 Lie theory

A *Lie algebra* $\mathfrak{g}$ over a field $\mathbb{F}$ is a linear space over $\mathbb{F}$ equipped with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called the *Lie bracket*, which alternates i.e. $[X, X] = 0$ for all $X \in \mathfrak{g}$, and satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. A Lie algebra is said to be *abelian* if $[\cdot, \cdot] \equiv 0$.

A trivial example of a Lie algebra is the n -dimensional abelian algebra $\mathfrak{t}^n$, which is isomorphic to $\mathbb{R}^n$. A slightly less trivial classical example is the space $\text{End}(V)$ of all endomorphisms of a linear space V equipped with the canonical commutator bracket. It is usually denoted by $\mathfrak{gl}(V)$. Another example is the algebra of skew-Hermitian matrices

$$\mathfrak{u}(n) = \{Z \in \mathbb{C}^{n \times n} \mid Z + Z^* = 0\}$$

equipped with the canonical commutator bracket.

A *derivation* on a Lie algebra $\mathfrak{g}$ is an endomorphism $D \in \text{End}(\mathfrak{g})$ such that

$$D[X, Y] = [DX, Y] + [X, DY],$$

for all $X, Y \in \mathfrak{g}$. The space of all derivations on $\mathfrak{g}$ is denoted by $\text{Der}(\mathfrak{g})$. This space becomes a Lie algebra when equipped with the canonical commutator bracket. An important example of a derivation is the *adjoint mapping* $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$ of an element $X \in \mathfrak{g}$, which is defined as

$$\text{ad}_X(Y) = [X, Y], \quad Y \in \mathfrak{g}.$$

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras and $\pi : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ is a Lie algebra homomorphism, then the *semidirect product* $\mathfrak{g} \ltimes_{\pi} \mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{h}$ with respect to π is the Lie algebra having $\mathfrak{g} \oplus \mathfrak{h}$ as the underlying linear space, and whose Lie bracket satisfies

$$[X, Y] = \pi(X)Y,$$

for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. When $\pi \equiv 0$, this simply becomes the natural *direct product* of Lie algebras.

For two linear subspaces $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$, we put

$$[\mathfrak{a}, \mathfrak{b}] = \{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\} \subset \mathfrak{g}.$$

If $\mathfrak{h} \subset \mathfrak{g}$ is a linear subspace, then we say that $\mathfrak{h}$ is a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, and an *ideal* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. Obviously every ideal of $\mathfrak{g}$ is also a subalgebra of $\mathfrak{g}$. Lie algebras are usually classified according to the following.

- (i) The *commutator series* $(\mathfrak{g}^k)_{k \geq 0}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k].$$

The Lie algebra $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^k = \{0\}$ for some k .

- (ii) The *lower central series* $(\mathfrak{g}_k)_{k \geq 0}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k].$$

The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if $\mathfrak{g}_k = \{0\}$ for some k .

(iii) The Lie algebra $\mathfrak{g}$ is said to be *simple* if $\mathfrak{g}$ is non-abelian and has no proper non-zero ideals.

(iv) The Lie algebra $\mathfrak{g}$ is said to be *semisimple* if it has no non-zero solvable ideals.

Note that each nilpotent Lie algebra is a fortiori solvable and each simple Lie algebra is a fortiori semisimple. Solvable and semisimple Lie algebras play an important role in Lie theory. One of the reasons for this is the famous Levi decomposition theorem, which roughly says that every finite-dimensional real Lie algebra is the semidirect product of a semisimple Lie algebra and a solvable Lie algebra.

It is often convenient to represent Lie algebras as algebras of linear transformations. A *linear representation* of $\mathfrak{g}$ on a linear space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. The representation ρ is said to be *faithful* if ρ is injective. We note that the underlying field of the linear space V need not coincide with the underlying field of the Lie algebra. For example, it is standard to represent the real Lie algebra $\mathfrak{u}(n)$ on the linear space $\mathbb{C}^n$, so that $\mathfrak{u}(n)$ becomes a real Lie algebra of complex matrices. Ado's theorem guarantees that every finite-dimensional Lie algebra (over a field of characteristic zero, e.g. $\mathbb{R}$ or $\mathbb{C}$) admits a faithful linear representation on some finite-dimensional linear space V . Since finite-dimensional linear spaces are unique up to isomorphism, we can always take the linear space V to be $\mathbb{R}^n$ or $\mathbb{C}^n$.

A notion that connects the algebraic nature of Lie theory with the geometric nature of the theory of manifolds is that of Lie groups. A *Lie group* is a group $(G, \cdot)$, where G is a smooth manifold such that the map

$$G \times G \ni (p, q) \mapsto p \cdot q^{-1} \in G$$

is smooth. A standard example of a Lie group is the space $\text{Aut}(V)$ of all automorphisms of a linear space V equipped with the composition operation. It is usually called the *general linear group* of V , denoted by $\mathbf{GL}(V)$. Another example is the *unitary group*

$$\mathbf{U}(n) = \{z \in \mathbb{C}^{n \times n} \mid z \cdot z^* = I\}$$

equipped with the matrix multiplication as its Lie group operation.

A map which is of particular importance in Lie group theory is the *left translation* by $p \in G$ i.e. the diffeomorphism $L_p : G \rightarrow G$ defined by $L_p(q) = pq$. A vector field X on G is said to be *left-invariant* if $X_{pq} = (dL_p)_q(X_q)$ for all $p, q \in G$.

Note that left-invariant vector fields are completely determined by their value at one point, e.g. the identity element $e \in G$. One can show that the commutator bracket $[X, Y] = XY - YX \in C^\infty(TM)$ of two left-invariant vector fields $X, Y \in C^\infty(TM)$ is again a left-invariant vector field. In particular, the study of left-invariant vector fields and their commutator brackets can be restricted to the study of tangents at the identity element $e \in G$. Thus, the tangent space $T_e G$ at the identity element $e \in G$ has a natural structure of a Lie algebra. This Lie algebra is called the *Lie algebra of the Lie group* G . For example, the Lie algebra of the general linear group $\mathbf{GL}(V)$ of a linear space V is $\mathfrak{gl}(V)$, and the Lie algebra of the unitary group $\mathbf{U}(n)$ is $\mathfrak{u}(n)$.

By Lie's third theorem, every real finite-dimensional Lie algebra is the Lie algebra of some connected simply connected Lie group G . Furthermore, one can show that if

two simply connected Lie groups have isomorphic Lie algebras, then the groups themselves are isomorphic. Thus, there is essentially a unique correspondence between real finite-dimensional Lie algebras and connected simply connected Lie groups.

One usually classifies Lie groups according to their Lie algebras, so that a Lie group G is said to be *solvable*, *nilpotent*, *simple*, *semisimple*, respectively, if its Lie algebra $\mathfrak{g}$ is. A large part of this work is devoted to solvable Lie groups. One can show that every connected simply connected solvable Lie group G of dimension n is diffeomorphic to $\mathbb{R}^n$. This means that we will have a global coordinate system for each solvable Lie group we work with.

If $(G, \cdot), (H, *)$ are Lie groups and $\mu : G \rightarrow \text{Aut}(H)$ is a smooth homomorphism, then the *semidirect product* $G \ltimes_{\mu} H$ of G and H with respect to μ is the Lie group having $G \times H$ as the underlying manifold, equipped with the group operation

$$(g, h)(g', h') = (g \cdot g', h * \mu(g)h').$$

We note that the identity element of $G \ltimes_{\mu} H$ is (e_G, e_H) , where e_G and e_H are the identity elements of G and H , respectively, and the inverse of an element $(g, h) \in G \ltimes_{\mu} H$ is given by

$$(g, h)^{-1} = (g^{-1}, \mu(g^{-1})h^{-1}).$$

One can show that the Lie algebra of a semidirect product $G \ltimes_{\mu} H$ is $\mathfrak{g} \ltimes_{d\tilde{\mu}} \mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H , respectively, and $\tilde{\mu} : g \mapsto \tilde{\mu}(g)$ denotes the differential of $h \mapsto \mu(g, h)$ evaluated at the identity. Conversely, given a Lie algebra homomorphism $\pi : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ and assuming that G and H are connected simply connected Lie groups, there is a unique $\mu : G \rightarrow \text{Aut}(H)$ such that $G \ltimes_{\mu} H$ is a connected simply connected Lie group with Lie algebra $\mathfrak{g} \ltimes_{\pi} \mathfrak{h}$. These facts combined with the aforementioned correspondence between Lie algebras and Lie groups imply that there is essentially a unique correspondence between (real finite-dimensional) semidirect products of Lie algebras and semidirect products of connected simply connected Lie groups.

It is convenient to represent Lie groups as matrix groups when possible. Analogously to representations of Lie algebras, a *linear representation* of a Lie group G on a linear space V is a smooth group homomorphism $\Pi : G \rightarrow \mathbf{GL}(V)$. The representation Π is said to be *faithful* if Π is injective. A faithful representation $\Pi : G \rightarrow \mathbf{GL}(V)$ of G on V induces the faithful representation $d\Pi_e : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$ on V . However, it should be noted that a Lie group might not have a faithful linear representation even if its Lie algebra does. The classical counterexample for this is the universal covering of the special linear group $\mathbf{SL}_2(\mathbb{R})$.

2.3 Riemannian geometry

A *Riemannian metric* g on a smooth manifold M is a smooth section of the bundle $T^*M \otimes T^*M$ assigning to each point $p \in M$ an inner product on the tangent space T_pM . In local coordinates $(x_1, \dots, x_n)$ on M , we have

$$g = \sum_{i,j=1}^n g_{ij} dx_i \otimes dx_j, \quad g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right),$$

where the components g_{ij} form a symmetric positive-definite matrix which is usually also denoted by g , by an abuse of notation.

A Riemannian metric g enables us to define some classical differential operators on (M, g) as follows.

- (i) The *gradient* of a function $\phi \in C^1(M, \mathbb{R})$ is the vector field $\text{grad } \phi$ characterized by

$$g(\text{grad } \phi, X) = d\phi(X) = X(\phi), \quad X \in C^\infty(TM).$$

- (ii) The *divergence* of a vector field $X \in C^1(TM)$ is defined as the trace of the linear map $\nabla X : Y \mapsto \nabla_Y X$, i.e.

$$\text{div } X = \text{Tr}_g \nabla X = \sum_{i=1}^n g(\nabla_{X_i} X, X_i),$$

where $X_1, \dots, X_n$ is a local orthonormal frame of TM , and ∇ is the Levi-Civita connection of (M, g) .

- (iii) The *Laplace-Beltrami operator* (or *tension field*) τ is the linear differential operator defined for $\phi \in C^2(M, \mathbb{R})$ by

$$\tau(\phi) = \text{div grad } \phi.$$

- (iv) The *conformality operator* κ is the bilinear differential operator defined for two functions $\phi, \psi \in C^1(M, \mathbb{R})$ by

$$\kappa(\phi, \psi) = g(\text{grad } \phi, \text{grad } \psi).$$

The definitions above only concern real-valued functions and vector fields. However, for our purposes it will be more convenient to work with functions and vector fields which take values in $\mathbb{C}$. In view of this, we complexify the tangent bundle by setting $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$, and we extend the metric to a complex bilinear form. The definitions of the differential operators presented above then extend naturally to complex-valued functions and vector field by requiring that the resulting operators become complex linear (resp. bilinear). The gradient of a complex-valued function $\phi : M \rightarrow \mathbb{C}$ then becomes a section of the bundle $T^{\mathbb{C}}M$, its tension field $\tau(\phi)$ becomes a complex-valued function on M , etc.

If $X_1, \dots, X_n$ is a local orthonormal frame of TM , then one easily sees from the definitions that

$$\tau(\phi) = \sum_{i=1}^n (X_i^2(\phi) - \nabla_{X_i} X_i(\phi)), \quad \kappa(\phi, \psi) = \sum_{i=1}^n X_i(\phi) X_i(\psi). \quad (2.1)$$

Alternatively, if $(x_1, \dots, x_n)$ is a local coordinate system on M , we have the local representations

$$\tau(\phi) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det g} \frac{\partial \phi}{\partial x_j} \right), \quad \kappa(\phi, \psi) = \sum_{i,j=1}^n g^{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j}, \quad (2.2)$$

where g^{ij} denotes the (i, j) -th component of the *cometric*, which can be identified with the inverse of the matrix (g_{ij}) .

We note that the Laplace-Beltrami operator satisfies the product rule

$$\tau(\phi\psi) = \psi \tau(\phi) + 2\kappa(\phi, \psi) + \phi \tau(\psi), \quad \phi, \psi \in C^2(M, \mathbb{C}).$$

Furthermore, if f is a holomorphic function defined on an open set containing the image $\phi(M) \subset \mathbb{C}$ of $\phi \in C^2(M, \mathbb{C})$, then we have the chain rule formula

$$\tau(f \circ \phi) = \kappa(\phi, \phi) f''(\phi) + \tau(\phi) f'(\phi).$$

If the function ϕ is real-valued, then it suffices to assume that f is a C^2 -function defined on an open subset of $\mathbb{R}$ containing the image $\phi(M)$.

Finally, let us recall some aspects of Riemannian Lie theory. A Riemannian metric g on G is said to be *left-invariant* if

$$g_q(X_q, Y_q) = g_{pq}((dL_p)_q(X_q), (dL_p)_q(Y_q)),$$

for all $p, q \in M$ and $X_q, Y_q \in T_q G$. A Lie group G equipped with a left-invariant Riemannian metric is said to be a *Riemannian Lie group*. There are several canonical ways of equipping a Lie group with a left-invariant metric. In what follows we discuss two ways which are used in this work. For simplicity we focus only on matrix Lie groups i.e. Lie subgroups $G \subset \mathbf{GL}_n(\mathbb{C})$. Note that a left-invariant metric is completely determined by its values at a point. Hence, to define a left-invariant metric on G , it suffices to choose an inner product g_e on the Lie algebra $\mathfrak{g}$, since the metric is by left-invariance determined at other points by

$$g_p(X_p, Y_p) = g_e((dL_{p^{-1}})_p(X_p), (dL_{p^{-1}})_p(Y_p)) = g_e(p^{-1}X_p, p^{-1}Y_p).$$

for all $p \in M$ and $X_p, Y_p \in T_p G$.

Possibly the most natural choice of inner product on the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ is the canonical inner product

$$g(Z, W) = \Re \operatorname{Tr}(ZW^*), \quad Z, W \in \mathfrak{g}.$$

We can obtain a simplified formula for the Laplace-Beltrami operator with respect to this metric. The Koszul formula for the Levi-Civita connection ∇ on $\mathbf{GL}_n(\mathbb{C})$ yields

$$g(\nabla_Z Z, W) = g([W, Z], Z) = g([Z, Z^*], W).$$

Since the Levi-Civita connection on G is simply the component of $\nabla_Z Z$ which is tangential to G , we see from the calculation above together with (2.1) that the Laplace-Beltrami operator with respect to this metric satisfies

$$\tau(\phi) = \sum_{Z \in \mathcal{B}} \{Z^2(\phi) - [Z, Z^*]_{\mathfrak{g}}(\phi)\},$$

where $\mathcal{B}$ is an orthonormal basis for $\mathfrak{g}$ and $[Z, Z^*]_{\mathfrak{g}}$ denotes the orthogonal projection of $[Z, Z^*]$ onto $\mathfrak{g}$. This becomes particularly convenient when the orthonormal basis

$\mathcal{B}$ of $\mathfrak{g}$ can be chosen so that each of its elements $Z \in \mathcal{B}$ is either Hermitian or skew-Hermitian, since in this case $[Z, Z^*] = 0$, and the formula for τ simplifies to

$$\tau(\phi) = \sum_{Z \in \mathcal{B}} Z^2(\phi).$$

On the other hand, when one is working in a coordinate system $x = (x_1, \dots, x_n)$ around the identity element $e \in G$, there is another more natural choice of inner product on $\mathfrak{g}$. We assume without loss of generality that $0 \in \mathbb{R}^n$ corresponds to $e \in G$ in this system. Then we see that the partial derivatives

$$X_i = \left. \frac{\partial}{\partial x_i} \right|_0 \in \mathfrak{g}$$

form a basis for $\mathfrak{g}$. In this case it is natural to define an inner product on $\mathfrak{g}$ by requiring that this basis becomes orthonormal on $\mathfrak{g}$ i.e. that

$$g(X_i, X_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. In this case we can obtain a nice formula for the resulting left-invariant metric in the coordinates $(x_1, \dots, x_n)$. Throughout the calculations we let $p \in G$ correspond to the point $x = (x_1, \dots, x_n)$ in the coordinate system. For $1 \leq i \leq n$ we can write

$$p^{-1} \left. \frac{\partial}{\partial x_i} \right|_p = \sum_{k=1}^n M_{ki} X_k,$$

for some smooth coefficients $M_{ki} = M(x)_{ki}$ which form an $n \times n$ matrix $M = M(x)$. Then we see that

$$\begin{aligned} g\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p\right) &= g\left(p^{-1} \left.\frac{\partial}{\partial x_i}\right|_p, p^{-1} \left.\frac{\partial}{\partial x_j}\right|_p\right) \\ &= \sum_{k,l=1}^n M_{ki} M_{lj} g(X_k, X_l) \\ &= \sum_{k=1}^n M_{ki} M_{kj} = (M^T M)_{ij}, \end{aligned}$$

so that the resulting left-invariant metric in the coordinates $x = (x_1, \dots, x_n)$ is given simply by

$$g = M(x)^T M(x).$$

Chapter 3

p -Harmonic Functions on Riemannian Manifolds

In this chapter we introduce the notion of p -harmonic functions on Riemannian manifolds. We then present a general method of constructing p -harmonic functions using isoparametric functions. This method will then be used extensively throughout the thesis to generate new examples of p -harmonic functions.

3.1 p -Harmonic functions

As discussed in the introduction, the Laplace-Beltrami operator provides us with a natural generalization of the classical Laplacian to Riemannian manifolds. The classical p -harmonic equation

$$\Delta^p \phi = 0$$

therefore also generalizes naturally to the Riemannian setting.

Definition 3.1.1. Let (M, g) be a Riemannian manifold and let p be a positive integer. The iterated Laplace-Beltrami operator τ^p is the linear differential operator defined inductively by

$$\tau^0(\phi) = \phi, \quad \tau^p(\phi) = \tau(\tau^{p-1}(\phi)),$$

for $\phi \in C^{2p}(M, \mathbb{C})$. The function ϕ is said to be p -harmonic if

$$\tau^p(\phi) = 0.$$

If ϕ is p -harmonic but not $(p-1)$ -harmonic, then ϕ is said to be *proper p -harmonic*.

Remark 3.1.2. Since the Laplace-Beltrami operator τ is an elliptic differential operator, it follows that τ^p is as well, so that any p -harmonic function is automatically smooth.

Some authors prefer to call such functions *polyharmonic* of order p . We note that 1-harmonic functions coincide with *harmonic* functions. The functions which are 2-harmonic are usually called *biharmonic*, 3-harmonic functions are called *triharmonic* etc. Observe that a function ϕ is proper p -harmonic if and only if $\tau(\phi)$ is proper

$(p-1)$ -harmonic, in other words, if and only if $\tau^{p-1}(\phi)$ is a non-zero harmonic function. Finally, let us observe that if a function ϕ is p -harmonic, then it is also r -harmonic for any $r \geq p$. Hence, one is usually interested in finding functions which are *proper* p -harmonic.

3.2 Isoparametric functions and p -harmonic functions

In this section we develop our workhorse for constructing p -harmonic functions on Riemannian manifolds. This method relies heavily on the existence of the following type of functions.

Definition 3.2.1. Let (M, g) be a Riemannian manifold. A smooth complex-valued function $\phi : M \rightarrow \mathbb{C}$ is said to be *isoparametric* on M if there exist holomorphic functions $\Phi, \Psi : U \rightarrow \mathbb{C}$ defined on some open set $U \subset \mathbb{C}$ containing $\phi(M)$, such that the tension field τ and the conformality operator κ satisfy

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad \kappa(\phi, \phi) = \Psi \circ \phi.$$

Classically, isoparametric functions are assumed to be real-valued, and weaker regularity assumptions such as continuity or smoothness are imposed on the functions Φ and Ψ . This class of functions has been extensively studied due to their beautiful geometric properties, see e.g. [43, 42, 38]. In fact, it has been shown in [43, 6] that in the classical case, a real-valued function is isoparametric if and only if its regular level sets are parallel and have constant mean curvatures. Complex-valued isoparametric functions have not been considered in such great detail to our knowledge, the only exception being the paper [4], in which the author studies complex-valued isoparametric functions on $\mathbb{R}^3$, though without requiring that Φ and Ψ are holomorphic. Our Definition 3.2.1 therefore seems to be new.

Isoparametric functions have already been considered in the context of biharmonic analysis on Riemannian manifolds in the paper [6]. In it, the authors observe that if $\phi : M \rightarrow \mathbb{R}$ is isoparametric, then the chain rule gives

$$\tau^2(\phi) = \tau(\Phi \circ \phi) = \kappa(\phi, \phi) \Phi''(\phi) + \tau(\phi) \Phi'(\phi) = (\Psi \Phi'' + \Phi \Phi') \circ \phi.$$

As a consequence, they conclude that an isoparametric function is biharmonic if and only if the functions Φ, Ψ satisfy the differential equation

$$\Psi \Phi'' + \Phi \Phi' = 0.$$

This observation allows them to demonstrate how one can deform the metric g on the Riemannian manifold M to a new metric $\tilde{g}$ with respect to which the isoparametric function ϕ is biharmonic. The method that we are about to present is similar in nature, but the upshot is that we are able to construct proper p -harmonic functions with respect to a fixed metric g , regardless of how the functions Φ and Ψ are related.

Suppose that $\phi : M \rightarrow \mathbb{C}$ is a complex-valued isoparametric function in the sense of Definition 3.2.1. The idea is to first construct a harmonic function of the form $f_1 \circ \phi$, where f_1 is holomorphic, and then to inductively construct proper p -harmonic functions of the form $f_p \circ \phi$, where f_p is holomorphic, by requiring that

$$\tau(f_p \circ \phi) = f_{p-1} \circ \phi, \quad p > 1. \tag{3.1}$$

The chain rule implies that

$$\tau(f_p \circ \phi) = \kappa(\phi, \phi) f_p''(\phi) + \tau(\phi) f_p'(\phi) = (\Psi f_p'' + \Phi f_p') \circ \phi,$$

so that (3.1) is equivalent to the elementary complex ordinary differential equation

$$\Psi f_p'' + \Phi f_p' = f_{p-1},$$

which can easily be solved. More explicitly, this results in the following.

Theorem 3.2.2. *Let (M, g) be a Riemannian manifold, and let $\phi : M \rightarrow \mathbb{C}$ be a complex-valued isoparametric function on M with*

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad \kappa(\phi, \phi) = \Psi \circ \phi,$$

for some holomorphic functions $\Phi, \Psi : U \rightarrow \mathbb{C}$ defined on some open set $U \subset \mathbb{C}$ containing $\phi(M)$. Suppose that one of the following situations holds.

- (i) *If Ψ vanishes identically, let $\hat{U}$ be an open simply connected subset of $U \setminus \Phi^{-1}(\{0\})$, and define the holomorphic functions $f_p : \hat{U} \rightarrow \mathbb{C}$ for $p \geq 1$ by*

$$f_p(z) = c \left(\int^z \frac{d\zeta}{\Phi(\zeta)} \right)^{p-1},$$

where $c \in \mathbb{C} \setminus \{0\}$.

- (ii) *If Ψ does not vanish identically, let $\hat{U}$ be an open simply connected subset of $U \setminus \Psi^{-1}(\{0\})$, put*

$$\Lambda(z) = \exp \left(- \int^z \frac{\Phi(\zeta)}{\Psi(\zeta)} d\zeta \right), \quad z \in \hat{U},$$

and define the holomorphic functions $f_p : \hat{U} \rightarrow \mathbb{C}$ for $p \geq 1$ by

$$\begin{aligned} f_1(z) &= c_1 \int^z \Lambda(\zeta) d\zeta + c_2, \\ f_p(z) &= \int^z \Lambda(\eta) \int^\eta \frac{f_{p-1}(\zeta)}{\Lambda(\zeta) \Psi(\zeta)} d\zeta d\eta, \quad p > 1, \end{aligned}$$

where $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$.

Then in both cases, the composition

$$f_p \circ \phi : \phi^{-1}(\hat{U}) \rightarrow \mathbb{C}$$

is proper p -harmonic on its open domain $\phi^{-1}(\hat{U})$ in M for all $p \geq 1$.

Remark 3.2.3. If the isoparametric function ϕ is real-valued, then one can weaken the assumptions in Theorem 3.2.2 by only requiring that Φ and Ψ are smooth functions of a real variable, and $\hat{U}$ can be taken to be subset of the real numbers which need not be simply connected. The requirement that $\hat{U}$ is simply connected in the complex-valued case is needed to ensure that the holomorphic antiderivatives are well-defined.

Proof of Theorem 3.2.2. The result can be obtained by solving the differential equations described before the statement of the theorem. For simplicity, we verify the result by direct differentiation.

Suppose first that Ψ vanishes identically. Then for any $p > 1$, we see that

$$f'_p(z) = (p-1) \frac{c}{\Phi(z)} \left(\int^z \frac{d\zeta}{\Phi(\zeta)} \right)^{p-2} = (p-1) \frac{f_p(z)}{\Phi(z)}.$$

Thus, the chain rule for the Laplace-Beltrami operator gives

$$\begin{aligned} \tau(f_p \circ \phi) &= \kappa(\phi, \phi) f''_p(\phi) + \tau(\phi) f'_p(\phi) \\ &= \Psi(\phi) f''_p(\phi) + \Phi(\phi) f'_p(\phi) = (p-1) f_{p-1}(\phi), \end{aligned}$$

and the result follows immediately by induction since f_1 is a non-zero constant.

Now suppose that Ψ does not vanish identically. Then we observe that

$$f'_1(z) = \Lambda(z) \quad \text{and} \quad f''_1(z) = \Lambda'(z) = -\frac{\Phi(z)}{\Psi(z)} \Lambda(z),$$

so that the chain rule for the Laplace-Beltrami operator gives

$$\tau(f_1 \circ \phi) = \Psi(\phi) f''_1(\phi) + \Phi(\phi) f'_1(\phi) = -\Psi(\phi) \frac{\Phi(z)}{\Psi(z)} \Lambda(\phi) + \Phi(\phi) \Lambda(\phi) = 0,$$

which shows that $f_1 \circ \phi$ is indeed proper harmonic on M . Now observe that, for $p > 1$, we have

$$\begin{aligned} f'_p(z) &= \Lambda(z) \int^z \frac{f_{p-1}(\zeta)}{\Lambda(\zeta) \Psi(\zeta)} d\zeta, \quad \text{and} \\ f''_p(z) &= \Lambda'(z) \int^z \frac{f_{p-1}(\zeta)}{\Lambda(\zeta) \Psi(\zeta)} d\zeta + \Lambda(z) \frac{f_{p-1}(z)}{\Lambda(z) \Psi(z)} \\ &= -\frac{\Phi(z)}{\Psi(z)} \Lambda(z) \int^z \frac{f_{p-1}(\zeta)}{\Lambda(\zeta) \Psi(\zeta)} d\zeta + \frac{f_{p-1}(z)}{\Psi(z)}. \end{aligned}$$

It follows by the chain rule that

$$\begin{aligned} \tau(f_p \circ \phi) &= \Psi(\phi) f''_p(\phi) + \Phi(\phi) f'_p(\phi) \\ &= -\Phi(\phi) \Lambda(\phi) \int^\phi \frac{f_{p-1}(\zeta)}{\Lambda(\zeta) \Psi(\zeta)} d\zeta + f_{p-1}(\phi) + \Phi(\phi) \Lambda(\phi) \int^\phi \frac{f_{p-1}(\zeta)}{\Lambda(\zeta) \Psi(\zeta)} d\zeta \\ &= f_{p-1}(\phi), \end{aligned}$$

from which the result follows by induction, since $f_1 \circ \phi$ is proper harmonic. $\square$

The functions f_p in case (ii) of Theorem 3.2.2 do not seem to have a nice closed form in general i.e. it does not seem that one can avoid defining them inductively. However, as we will see throughout this thesis, it often turns out that the functions furnished by the theorem are of the form

$$x \mapsto f_p(\phi(x)) = h(\phi(x))^{p-1} f_1(\phi(x)) + (p-1)\text{-harmonic terms},$$

for some holomorphic function h . The $(p-1)$ -harmonic terms will usually be discarded, for simplicity.

Example 3.2.4. To see Theorem 3.2.2 in action, let us use it to construct proper p -harmonic functions on some classical Riemannian manifolds. We consider manifolds M which are open subsets of $\mathbb{R}^2 \setminus \{0\}$ equipped with the radially symmetric Riemannian metric

$$g_\rho = \frac{dx \otimes dx + dy \otimes dy}{\rho(\sqrt{x^2 + y^2})},$$

where ρ is a smooth positive function defined on $(0, \infty)$. Note that these manifolds include (subsets of) the classical two-dimensional manifolds of constant curvature. Indeed:

- (i) If $\rho \equiv 1$ and $M = \mathbb{R}^2 \setminus \{0\}$, this is simply the flat two-dimensional Euclidean space minus the origin;
- (ii) If $\rho(s) = (1 + s^2)^2/4$ and $M = \mathbb{R}^2 \setminus \{0\}$, we obtain the two-dimensional *punctured sphere* minus the origin;
- (iii) If $\rho(s) = (1 - s^2)^2/4$ and $M = \mathbb{D}_1 \setminus \{0\}$ where $\mathbb{D}_1$ denotes the unit disk, we obtain the two-dimensional *hyperbolic disk* minus the origin.

It is easy to see that the Laplace-Beltrami operator and the conformality operator on the Riemannian manifold (M, g_ρ) satisfy

$$\tau(\phi) = \rho(\sqrt{x^2 + y^2}) \Delta \phi, \quad \kappa(\phi, \psi) = \rho(\sqrt{x^2 + y^2}) \langle \nabla \phi, \nabla \psi \rangle,$$

where Δ and ∇ are the Laplacian and the gradient of flat $\mathbb{R}^2$, and $\langle \cdot, \cdot \rangle$ denotes the canonical inner product of $\mathbb{R}^2$.

Now consider the real-valued function $\phi : M \rightarrow \mathbb{R}$ given by

$$\phi(x, y) = \sqrt{x^2 + y^2},$$

Then an elementary calculation shows that

$$\tau(\phi) = \frac{\rho(\phi)}{\phi} \quad \text{and} \quad \kappa(\phi, \phi) = \rho(\phi),$$

so that ϕ is isoparametric on M . This unlocks Theorem 3.2.2, which we will now use to construct a class of proper p -harmonic functions on M .

Since ϕ is real-valued, we may assume that the functions from the Theorem 3.2.2 are functions of one real variable. In the notation of Theorem 3.2.2, we have

$$\Phi(s) = \frac{\rho(s)}{s} \quad \text{and} \quad \Psi(s) = \rho(s),$$

and we see that we may take the set $\hat{U}$ to be the positive real axis $(0, \infty)$. The function Λ is given by

$$\Lambda(s) = \exp \left(- \int^s \frac{\rho(\zeta)}{\zeta \rho(\zeta)} d\zeta \right) = \exp(-\log(s)) = \frac{1}{s},$$

which yields

$$f_1(s) = c_1 \int^s \frac{d\zeta}{\zeta} + c_2 = c_1 \log(s) + c_2,$$

as well as

$$f_p(s) = \int^s \frac{1}{\eta} \int^\eta \frac{\zeta f_{p-1}(\zeta)}{\rho(\zeta)} d\zeta d\eta.$$

It therefore follows from Theorem 3.2.2 that the function

$$(x, y) \mapsto c_1 \log \sqrt{x^2 + y^2} + c_2$$

is proper harmonic on M , and

$$(x, y) \mapsto \int^{\sqrt{x^2+y^2}} \frac{1}{\eta} \int^\eta \frac{\zeta f_{p-1}(\zeta)}{\rho(\zeta)} d\zeta d\eta$$

is proper p -harmonic on M for $p > 1$.

If $\rho \equiv 1$, then one can compute that the proper p -harmonic function above is given by

$$(x, y) \mapsto c_1(x^2 + y^2)^{p-1} \log \sqrt{x^2 + y^2} + c_2(x^2 + y^2)^{p-1},$$

which is also well-known to be p -harmonic by Almansi's Theorem [3]. For the other two aforementioned cases, namely the punctured sphere and the hyperbolic disk, the functions f_p do not seem to have a simple closed form. However, if one sets $c_1 = 0, c_2 = 1$, then after a simple calculation of the integrals one obtains the biharmonic functions

$$(x, y) \mapsto \begin{cases} \log(1 + (x^2 + y^2)), & \text{if } \rho(s) = (1 + s^2)^2/4 \\ \log(1 - (x^2 + y^2)), & \text{if } \rho(s) = (1 - s^2)^2/4 \end{cases}.$$

That these functions are biharmonic on the corresponding Riemannian manifold (M, g_ρ) is well-known, see [19]. $\triangle$

Chapter 4

Eigenfunctions on Riemannian manifolds

This chapter is based on the paper [27]. We apply Theorem 3.2.2 to a particular class of isoparametric functions called eigenfunctions, and thus show how to construct proper p -harmonic functions using eigenfunctions. We then present examples of eigenfunctions on the important classical semisimple Lie groups $\mathbf{SL}_n(\mathbb{R})$, $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$, $\mathbf{SO}(p, q)$, $\mathbf{SU}(p, q)$, $\mathbf{Sp}(p, q)$, $\mathbf{Sp}(n, \mathbb{R})$, $\mathbf{SO}^*(2n)$, and $\mathbf{SU}^*(2n)$.

4.1 Eigenfunctions and p -harmonic functions

We start by defining eigenfunctions on Riemannian manifolds.

Definition 4.1.1. [27] Let (M, g) be a Riemannian manifold. A smooth complex-valued function $\phi : M \rightarrow \mathbb{C}$ is said to be an *eigenfunction* on M if there exist constants $\lambda, \mu \in \mathbb{C}$ with

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu\phi^2.$$

In particular, observe that every eigenfunction is a fortiori isoparametric, in the sense of Definition 3.2.1.

Example 4.1.2. Equip the manifold $M = \mathbb{R}^2 \setminus \{0\}$ with the radially symmetric Riemannian metric

$$g = \frac{dx \otimes dx + dy \otimes dy}{x^2 + y^2}.$$

Then the function $\phi : M \rightarrow \mathbb{R}$ defined by

$$\phi(x, y) = \sqrt{x^2 + y^2}$$

is an eigenfunction on M satisfying

$$\tau(\phi) = \phi \quad \text{and} \quad \kappa(\phi, \phi) = \phi^2. \quad \triangle$$

A closely related concept is that of *eigenfamilies*, namely collections of smooth complex-valued functions on M such that there exist constants $\lambda, \mu \in \mathbb{C}$ with

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu\phi\psi,$$

for all ϕ, ψ belonging to the collection. Clearly every element of an eigenfamily is an eigenfunction. Eigenfamilies were introduced in [23] by Gudmundsson and Sakovich. In the same paper, the authors construct examples of eigenfamilies on classical semisimple Lie groups. Eigenfamilies have turned out to be very useful for the construction of harmonic morphisms on Riemannian manifolds, see [23, 24]

We are now ready to state the main result of the paper [27].

Theorem 4.1.3. [27] *Let $\phi : (M, g) \rightarrow \mathbb{C}$ be a complex-valued eigenfunction on a Riemannian manifold such that the tension field τ and the conformality operator κ satisfy*

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu\phi^2$$

for some $\lambda, \mu \in \mathbb{C}$. Define the holomorphic functions $f_p : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ by

$$f_p(z) = \begin{cases} c_1 \log(z)^{p-1}, & \text{if } \mu = 0, \lambda \neq 0 \\ c_1 \log(z)^{2p-1} + c_2 \log(z)^{2p-2}, & \text{if } \mu \neq 0, \lambda = \mu \\ c_1 z^{1-\frac{\lambda}{\mu}} \log(z)^{p-1} + c_2 \log(z)^{p-1}, & \text{if } \mu \neq 0, \lambda \neq \mu \end{cases}$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary but such that f_p does not vanish identically. Then the composition

$$f_p \circ \phi : W = \{x \in M \mid \phi(x) \notin (-\infty, 0]\} \rightarrow \mathbb{C}$$

is proper p -harmonic on its open domain W in M .

Remark 4.1.4. Here $\log : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ is the holomorphic principal logarithm and

$$z^\alpha = \exp(\alpha \log(z)).$$

Proof. Throughout the proof, we use the notation as in Theorem 3.2.2. We see that the holomorphic functions $\Phi, \Psi : \mathbb{C} \rightarrow \mathbb{C}$ satisfy

$$\Phi(z) = \lambda z, \quad \Psi(z) = \mu z^2.$$

Furthermore, $\Phi^{-1}(\{0\}) = \Psi^{-1}(\{0\}) = \{0\}$, so we may select our simply connected domain $\hat{U}$ to be the slit plane $\mathbb{C} \setminus (-\infty, 0]$ in all three cases. To complete the proof of the result, we must only calculate the antiderivatives from Theorem 3.2.2.

If $\mu = 0$ and $\lambda \neq 0$, then Ψ vanishes identically, so that the holomorphic functions $f_p : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ are given by

$$f_p(z) = c \left(\int^z \frac{d\zeta}{\lambda \zeta} \right)^{p-1} = \frac{c}{\lambda^{p-1}} \log(z)^{p-1},$$

for an arbitrary constant $c \in \mathbb{C} \setminus \{0\}$, and the result follows.

Now consider the case $\mu \neq 0$ and $\mu = \lambda$. Then it follows that the function $\Lambda : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ is given by

$$\Lambda(z) = \exp(-\log(z)) = \frac{1}{z},$$

and so $f_1 : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ satisfies

$$f_1(z) = c_1 \int^z \frac{d\zeta}{\zeta} + c_2 = c_1 \log(z) + c_2,$$

for arbitrary $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$, which shows that the result holds for $p = 1$. For the induction step, we assume that f_p has the claimed form, in which case

$$\begin{aligned} f_{p+1}(z) &= \int^z \frac{1}{\eta} \int^\eta \frac{1}{\mu \zeta} (c_1 \log(\zeta)^{2p-1} + c_2 \log(\zeta)^{2p-2}) d\zeta d\eta \\ &= \frac{c_1}{2p\mu} \int^z \frac{1}{\eta} \log(\eta)^{2p} d\eta + \frac{c_2}{(2p-1)\mu} \int^z \frac{1}{\eta} \log(\eta)^{2p-1} d\eta \\ &= \frac{c_1}{2p(2p+1)\mu} \log(z)^{2p+1} + \frac{c_2}{2p(2p-1)\mu} \log(z)^{2p}, \end{aligned}$$

so that the induction principle gives the claim.

Now finally suppose that $\mu \neq 0$ and $\mu \neq \lambda$. The function $\Lambda : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ is then given by

$$\Lambda(z) = \exp\left(-\frac{\lambda}{\mu} \log(z)\right) = z^{-\frac{\lambda}{\mu}}.$$

Hence, the function $f_1 : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ satisfies

$$f_1(z) = c_1 \int^z \zeta^{-\frac{\lambda}{\mu}} d\zeta + c_2 = c_1 z^{1-\frac{\lambda}{\mu}} + c_2,$$

so that the result holds for $p = 1$. To prove the induction step, we will use the fact that, for any $\nu \in \mathbb{C} \setminus \{-1\}$ and non-negative integer n ,

$$\int^z \zeta^\nu \log(\zeta)^n d\zeta = z^{1+\nu} \sum_{j=0}^n a_j \log(z)^j,$$

for an appropriate choice of non-zero constants $a_j \in \mathbb{C}$. This follows easily by induction on the exponent n and integration by parts (alternatively, see item 2.722 of the useful collection [17]). Returning to our main proof, we proceed by strong induction i.e. we assume that the functions $f_1, \dots, f_p$ have the required form. Then

$$\begin{aligned} f_{p+1}(z) &= \int^z \eta^{-\frac{\lambda}{\mu}} \int^\eta \zeta^{\frac{\lambda}{\mu}-2} (c_1 \zeta^{1-\frac{\lambda}{\mu}} \log(\zeta)^{p-1} + c_2 \log(\zeta)^{p-1}) d\zeta d\eta \\ &= \frac{c_1}{p} \int^z \eta^{-\frac{\lambda}{\mu}} \log(\eta)^p d\eta + c_2 \sum_{j=0}^{p-1} a_j \int^z \frac{1}{\eta} \log(\eta)^j d\eta \\ &= \frac{c_1}{p} z^{1-\frac{\lambda}{\mu}} \sum_{i=0}^p b_i \log(z)^i + c_2 \sum_{j=0}^{p-1} \frac{a_j}{j+1} \log(z)^{j+1}, \end{aligned}$$

where the constants b_i, a_j are appropriately chosen. Now the terms with $i < p$ and $j < p-1$ will be p -harmonic after pulling back by ϕ by the induction hypothesis, so we may discard them for simplicity. Thus, after modifying the constants we obtain

$$f_{p+1}(z) = c_1 z^{1-\frac{\lambda}{\mu}} \log(z)^p + c_2 \log(z)^p,$$

which completes the proof. □

4.2 Eigenfunctions on classical semisimple Lie groups

In this section we will show that many of the classical semisimple Riemannian Lie subgroups of $\mathbf{GL}_n(\mathbb{C})$ admit proper p -harmonic functions. To achieve this we will construct eigenfunctions on these groups, since this unlocks the construction furnished by Theorem 4.1.3.

Let us recall the definitions of some of the classical semisimple Lie groups, before listing the eigenfunctions in Table 4.1. First, we fix our notation. We denote the $n \times n$ identity matrix by I_n and we also use the standard notations

$$J_{p,q} = \begin{bmatrix} 0 & I_p \\ -I_q & 0 \end{bmatrix}, \quad I_{p,q} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

In Table 4.1 and throughout this section, we interpret the vectors as row vectors, so that if $a, v \in \mathbb{C}^n$, then $a^T v$ is an $n \times n$ matrix with entries $a_i v_j$. We also put

$$\mathbb{C}_1^p = \{(z, w) \in \mathbb{C}^p \times \mathbb{C}^q \mid w = 0\}, \quad \mathbb{C}_2^q = \{(z, w) \in \mathbb{C}^p \times \mathbb{C}^q \mid z = 0\}.$$

The Lie groups in this section are all assumed to be equipped with the left-invariant Riemannian metric induced by the canonical inner product

$$g(Z, W) = \Re \operatorname{Tr}(ZW^*)$$

on the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. We will consider the well-known special linear group

$$\mathbf{SL}_n(\mathbb{R}) = \{x \in \mathbf{GL}_n(\mathbb{R}) \mid \det x = 1\}.$$

Moreover we will consider the important generalized orthogonal, unitary, and quaternionic unitary Lie groups $\mathbf{O}(p, q)$, $\mathbf{U}(p, q)$, $\mathbf{Sp}(p, q)$, which are defined as the groups of linear transformations on $\mathbb{R}^{p+q}$, $\mathbb{C}^{p+q}$, $\mathbb{H}^{p+q}$, respectively, preserving a bilinear (or Hermitian) form induced by $I_{p,q}$. More precisely,

$$\begin{aligned} \mathbf{O}(p, q) &= \{x \in \mathbf{GL}_{p+q}(\mathbb{R}) \mid x \cdot I_{p,q} \cdot x^T = I_{p,q}\}, \\ \mathbf{U}(p, q) &= \{z \in \mathbf{GL}_{p+q}(\mathbb{C}) \mid z \cdot I_{p,q} \cdot z^* = I_{p,q}\}, \\ \mathbf{Sp}(p, q) &= \{q \in \mathbf{GL}_{p+q}(\mathbb{H}) \mid q \cdot I_{p,q} \cdot q^* = I_{p,q}\}, \end{aligned}$$

and we also put

$$\begin{aligned} \mathbf{SO}(p, q) &= \mathbf{O}(p, q) \cap \mathbf{SL}_{p+q}(\mathbb{R}), \\ \mathbf{SU}(p, q) &= \mathbf{U}(p, q) \cap \mathbf{SL}_{p+q}(\mathbb{C}). \end{aligned}$$

We note that their classical counterparts $\mathbf{O}(n)$, $\mathbf{U}(n)$, $\mathbf{Sp}(n)$, $\mathbf{SO}(n)$, $\mathbf{SU}(n)$ can be seen as degenerate cases, in the sense that

$$\mathbf{O}(n) = \mathbf{O}(n, 0) = \mathbf{O}(0, n),$$

and analogously for the other groups. For the quaternionic unitary group $\mathbf{Sp}(p, q)$ we use the standard complex representation of $\mathbf{GL}_{p+q}(\mathbb{H})$ in $\mathbb{C}^{2(p+q) \times 2(p+q)}$ given by

$$(z + jw) \mapsto q = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.$$

We will also consider the symplectic group

$$\mathbf{Sp}(n, \mathbb{R}) = \{q \in \mathbf{SL}_{2n}(\mathbb{R}) \mid q \cdot J_{n,n} \cdot q^T = J_{n,n}\}.$$

Note that every element $q \in \mathbf{Sp}(n, \mathbb{R})$ can be represented as

$$q = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

for some matrices $x, y, z, w \in \mathbb{R}^{n \times n}$. Finally, we will consider the Lie groups

$$\begin{aligned} \mathbf{SO}^*(2n) &= \{q \in \mathbf{SU}(n, n) \mid q \cdot I_{n,n} \cdot J_{n,n} \cdot q^T = I_{n,n} \cdot J_{n,n}\}, \\ \mathbf{SU}^*(2n) &= \mathbf{U}^*(2n) \cap \mathbf{SL}_{2n}(\mathbb{C}), \end{aligned}$$

where

$$\mathbf{U}^*(2n) = \left\{ z + jw = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \mid z, w \in \mathbf{GL}_n(\mathbb{C}) \right\}.$$

Note that for any element $q \in \mathbf{SO}^*(2n)$, we have a unique representation

$$q = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix},$$

where z, w are $n \times n$ matrices.

Having recalled the definitions of these classical semisimple Lie groups, we are ready to present examples of eigenfunctions on these Lie groups. We gather these examples in Table 4.1. The rest of this section is devoted to studying these eigenfunctions and their tension field and conformality operator in more detail. Note that the eigenfunctions in Table 4.1 are homogeneous first degree polynomials of the coordinate functions of the Lie groups. More precisely, for a Lie group $G \subset \mathbf{GL}_n(\mathbb{C})$ the coordinate functions are given for $1 \leq i, j \leq n$ by

$$G \ni z \mapsto z_{ij} \in \mathbb{C}$$

where z_{ij} is the (i, j) -th entry of the matrix $z \in G$ expressed in the canonical orthonormal basis of $\mathbb{C}^n$, and the functions in consideration are of the form

$$\phi(z) = \sum_{i,j=1}^n A_{ij} z_{ij} = \text{Tr}(Az^T)$$

for some matrix $A \in \mathbb{C}^{n \times n}$. Observe that by the linearity of the Laplace-Beltrami operator we have

$$\tau(\phi) = \sum_{i,j=1}^n A_{ij} \tau(z_{ij}),$$

while by the bilinearity of the conformality operator we have

$$\kappa(\phi, \phi) = \sum_{i,j,k,l=1}^n A_{ij} A_{kl} \kappa(z_{ij}, z_{kl}).$$

Lie group	Eigenfunction ϕ	$\tau(\phi)$	$\kappa(\phi, \phi)$	Conditions
$\mathbf{SL}_n(\mathbb{R})$	$\text{Tr}(Ax^T)$	$\frac{n-1}{n} \phi$	$-\frac{1}{n} \phi^2$	$A \in \mathbb{C}^{n \times n}, AA^T = 0$
$\mathbf{SO}(n)$	$\text{Tr}(a^T vx^T)$	$-\frac{n-1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a, v \in \mathbb{C}^n, v$ isotropic
$\mathbf{SU}(n)$	$\text{Tr}(a^T vz^T)$	$-\frac{n^2-1}{n} \phi$	$-\frac{n-1}{n} \phi^2$	$a, v \in \mathbb{C}^n$
$\mathbf{Sp}(n)$	$\text{Tr}(a^T vz^T + a^T uw^T)$	$-\frac{2n+1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a, v, u \in \mathbb{C}^n$
$\mathbf{SO}(p, q)$	$\text{Tr}(a^T vx^T)$	$\frac{q-p+1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a \in \mathbb{C}^n, v \in \mathbb{C}_1^p, v$ isotropic
	$\text{Tr}(a^T vx^T)$	$\frac{p-q+1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a \in \mathbb{C}^n, v \in \mathbb{C}_2^q, v$ isotropic
$\mathbf{SU}(p, q)$	$\text{Tr}(a^T vz^T)$	$\frac{q^2-p^2+1}{p+q} \phi$	$-\frac{p+q-1}{p+q} \phi^2$	$a \in \mathbb{C}^n, v \in \mathbb{C}_1^p$
	$\text{Tr}(a^T vz^T)$	$\frac{p^2-q^2+1}{p+q} \phi$	$-\frac{p+q-1}{p+q} \phi^2$	$a \in \mathbb{C}^n, v \in \mathbb{C}_2^q$
$\mathbf{Sp}(p, q)$	$\text{Tr}(a^T vz^T + a^T uw^T)$	$-\frac{2(p-q)+1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a \in \mathbb{C}^n, v, u \in \mathbb{C}_1^p$
	$\text{Tr}(a^T vz^T + a^T uw^T)$	$-\frac{2(q-p)+1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a \in \mathbb{C}^n, v, u \in \mathbb{C}_2^q$
$\mathbf{Sp}(n, \mathbb{R})$	$\text{Tr}(a^T v(x + iy)^T)$	$\frac{1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a, v \in \mathbb{C}^n$
	$\text{Tr}(a^T v(z + iw)^T)$	$\frac{1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a, v \in \mathbb{C}^n$
$\mathbf{SO}^*(2n)$	$\text{Tr}(a^T vz^T)$	$-\frac{1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a, v \in \mathbb{C}^n$
	$\text{Tr}(a^T vw^T)$	$-\frac{1}{2} \phi$	$-\frac{1}{2} \phi^2$	$a, v \in \mathbb{C}^n$
$\mathbf{SU}^*(2n)$	$\text{Tr}(Az^T + Bw^T)$	$-\frac{2n+1}{2n} \phi$	$-\frac{1}{2n} \phi^2$	$A, B \in \mathbb{C}^{n \times n}, AB^T = BA^T$

Table 4.1: [27] Eigenfunctions on classical semisimple Lie groups.

Thus, to compute $\tau(\phi)$ and $\kappa(\phi, \phi)$, it suffices to calculate $\tau(z_{ij})$ and $\kappa(z_{ij}, z_{kl})$. Conveniently, the Lie algebra $\mathfrak{g}$ of each of our Lie groups admits an orthonormal basis $\mathcal{B}$ consisting of matrices which are either Hermitian or skew-Hermitian, which unlocks the simplified formula for the tension field, as discussed in Section 2.3. Note that if $Z \in \mathfrak{g}$, then

$$\begin{aligned}
Z(z_{ij}) &= \left. \frac{d}{dt} \right|_0 (z \text{Exp}(Zt))_{ij} = (zZ)_{ij}, \\
Z^2(z_{ij}) &= \left. \frac{d^2}{dt^2} \right|_0 (z \text{Exp}(Zt))_{ij} = (zZ^2)_{ij}.
\end{aligned}$$

In particular, we see that for $1 \leq i, j, k, l \leq n$, the Laplace-Beltrami operator on G satisfies

$$\tau(z_{ij}) = \sum_{Z \in \mathcal{B}} Z^2(z_{ij}) = \left(z \sum_{Z \in \mathcal{B}} Z^2 \right)_{ij},$$

and the conformality operator satisfies

$$\kappa(z_{ij}, z_{kl}) = \sum_{Z \in \mathcal{B}} Z(z_{ij}) Z(z_{kl}) = \sum_{Z \in \mathcal{B}} (zZ)_{ij} (zZ)_{kl} = \left(z \left\{ \sum_{Z \in \mathcal{B}} Z E_{jl} Z^T \right\} z^T \right)_{ik},$$

where E_{jl} denotes the $n \times n$ matrix satisfying

$$(E_{jl})_{\alpha\beta} = \delta_{j\alpha} \delta_{l\beta},$$

i.e. the matrix E_{jl} has 1 on the (j, l) -th position and zeroes elsewhere. Thus, the calculations of the tension field and the conformality operator of the coordinate functions of the Lie group reduce to the computation of the sums

$$\sum_{Z \in \mathcal{B}} Z^2 \quad \text{and} \quad \sum_{Z \in \mathcal{B}} Z E_{jl} Z^T.$$

Let us demonstrate some details of these calculations for the Lie group $\mathbf{SU}(n)$. The calculations for the other Lie groups of course follow the same pattern, see [23, 24] for the details. Note that, to calculate the tension field and the conformality operator of the coordinate functions on $\mathbf{SU}(n)$, it suffices to calculate them on $\mathbf{U}(n)$. To see this, we let τ, κ and $\hat{\tau}, \hat{\kappa}$ denote the tension field and the conformality operator of $\mathbf{SU}(n)$ and $\mathbf{U}(n)$, respectively. The matrix

$$W = \frac{i}{\sqrt{n}} I_n \in \mathfrak{u}(n)$$

is of unit length and generates the orthogonal complement of $\mathfrak{su}(n)$ in $\mathfrak{u}(n)$. Thus, if $\mathcal{B}$ denotes any orthonormal basis for $\mathfrak{su}(n)$, then $\hat{\mathcal{B}} = \mathcal{B} \cup \{W\}$ is an orthonormal basis for $\mathfrak{u}(n)$. Given any two smooth functions $\phi, \psi : \mathbf{U}(n) \rightarrow \mathbb{C}$, we therefore see that their restrictions to $\mathbf{SU}(n)$ (which we denote by the same letters) satisfy

$$\tau(\phi) = \sum_{Z \in \mathcal{B}} Z^2(\phi) = \sum_{Z \in \hat{\mathcal{B}}} Z^2(\phi) - W^2(\phi) = \hat{\tau}(\phi) - W^2(\phi),$$

and

$$\begin{aligned} \kappa(\phi, \psi) &= \sum_{Z \in \mathcal{B}} Z(\phi) Z(\psi) = \sum_{Z \in \hat{\mathcal{B}}} Z(\phi) Z(\psi) - W(\phi) W(\psi) \\ &= \hat{\kappa}(\phi, \psi) - W(\phi) W(\psi). \end{aligned}$$

In particular, if $\phi(z) = z_{ij}$ and $\psi(z) = z_{kl}$ for $1 \leq i, j, k, l \leq n$, these relations become

$$\tau(z_{ij}) = \hat{\tau}(z_{ij}) + \frac{1}{n} z_{ij}, \quad \text{and} \quad \kappa(z_{ij}, z_{kl}) = \hat{\kappa}(z_{ij}, z_{kl}) + \frac{1}{n} z_{ij} z_{kl}. \quad (4.1)$$

Now to calculate $\hat{\tau}(z_{ij})$ and $\hat{\kappa}(z_{ij}, z_{kl})$, we recall that the Lie algebra of $\mathbf{U}(n)$ is given by

$$\mathfrak{u}(n) = \{Z \in \mathfrak{gl}_n(\mathbb{C}) \mid Z + Z^* = 0\}.$$

In order to construct an orthonormal basis for this Lie algebra, we define the $n \times n$ matrices

$$X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}), \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}), \quad D_t = E_{tt},$$

for $1 \leq r < s \leq n$ and $1 \leq t \leq n$. Note that the matrices X_{rs} and D_t are Hermitian, and the matrices Y_{rs} are skew-Hermitian. These matrices induce the canonical orthonormal basis for the Lie algebra $\mathfrak{u}(n)$ given by

$$\hat{\mathcal{B}} = \{Y_{rs}, iX_{rs} \mid 1 \leq r < s \leq n\} \cup \{iD_t \mid 1 \leq t \leq n\}.$$

Using the identities discussed in Appendix A.1, we thus see that

$$\sum_{Z \in \hat{\mathcal{B}}} Z^2 = \sum_{1 \leq r < s \leq n} Y_{rs}^2 - \sum_{1 \leq r < s \leq n} X_{rs}^2 - \sum_{t=1}^n D_t^2 = -n I_n,$$

and

$$\sum_{Z \in \hat{\mathcal{B}}} Z E_{jl} Z^T = \sum_{1 \leq r < s \leq n} Y_{rs} E_{jl} Y_{rs}^T - \sum_{1 \leq r < s \leq n} X_{rs} E_{jl} X_{rs}^T - \sum_{t=1}^n D_t E_{jl} D_t^T = -E_{lj}.$$

It follows that, on $\mathbf{U}(n)$, we have

$$\hat{\tau}(z_{ij}) = -n z_{ij}, \quad \text{and} \quad \hat{\kappa}(z_{ij}, z_{kl}) = -(z E_{lj} z^T)_{ik} = -z_{il} z_{kj}.$$

In view of (4.1), this implies that on $\mathbf{SU}(n)$,

$$\tau(z_{ij}) = -\frac{n^2 - 1}{n} z_{ij}, \quad \text{and} \quad \kappa(z_{ij}, z_{kl}) = -z_{il} z_{kj} + \frac{1}{n} z_{ij} z_{kl}.$$

Finally, to obtain the relations from Table 4.1 for the case of $\mathbf{SU}(n)$, we consider the corresponding function $\phi : \mathbf{SU}(n) \rightarrow \mathbb{C}$ given by

$$\phi(z) = \text{Tr}(a^T v z^T) = \sum_{i,j=1}^n a_i v_j z_{ij},$$

where $a, v \in \mathbb{C}$. Then we obtain

$$\begin{aligned} \tau(\phi) &= \sum_{ij} a_i v_j \tau(z_{ij}) = -n \sum_{ij} a_i v_j z_{ij} = -n\phi, \\ \kappa(\phi, \phi) &= \sum_{ijkl} a_i v_j a_k v_l \kappa(z_{ij}, z_{kl}) \\ &= - \sum_{il} a_i v_l z_{il} \sum_{jk} a_j v_k z_{jk} + \frac{1}{n} \sum_{ij} a_i v_j z_{ij} \sum_{kl} a_k v_l z_{kl} \\ &= -\phi^2 + \frac{1}{n} \phi^2 = -\frac{n-1}{n} \phi^2, \end{aligned}$$

which confirms the result.

Chapter 5

Semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

In this chapter we study the Lie groups $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$, where $\mathbb{H}^{2n+1}$ denotes the $(2n+1)$ -dimensional Heisenberg group. We start the chapter by motivating our interest in studying such Lie groups. We then provide linear representations of such semidirect products and calculate the canonical left-invariant Riemannian metric for the respective representations, cf. end of Section 2.3. Finally, we construct p -harmonic functions on these Lie groups.

Throughout this chapter, we will denote by $\mathfrak{r}^n$ the n -dimensional abelian Lie algebra i.e. the Lie algebra of the Lie group $(\mathbb{R}^n, +)$.

5.1 Low-dimensional Lie algebras and Lie groups

Low-dimensional Lie groups, particularly of dimension three and four, are of great importance in physics. Probably most notably, such Lie groups are used as models of spacetime in the theory of general relativity. It is therefore not surprising that the low-dimensional Lie groups have already been studied in the context of (semi-)Riemannian geometry, see for example the papers [2, 36, 10]. Conveniently, low-dimensional connected simply connected Lie groups can be classified into disjoint non-isomorphic classes. As there is a bijection between real Lie algebras and connected simply connected Lie groups, this amounts to classifying the real Lie algebras of the corresponding dimension. Bianchi classified the three-dimensional real Lie algebras in his 1918 paper [8]. Nearly half a century later in 1963, Mubarakzyanov presented the first complete and correct classification of the four-dimensional Lie algebras in his work [35]. Today, the classification of real Lie algebras of dimension less than or equal to six is well-known, and can be found in the recent book [41]. The reader interested in more details about the history of the classifications can find them in [37].

The study conducted in this thesis was initially meant to be a study of p -harmonic functions on the four-dimensional connected simply connected Lie groups. For our purposes, only the Lie groups whose Lie algebras are indecomposable i.e. not direct products of lower dimensional Lie algebras, are of interest. The reason for this is that the Lie groups whose Lie algebras are decomposable are themselves direct products of lower dimensional Lie groups, and the theory of p -harmonic functions on product

manifolds is well-known, see e.g. [19]. Now let us briefly analyze the three- and four-dimensional real Lie algebras. The classifications that we use can be found e.g. in the papers [37, 9], and are presented here in Tables 5.1 and 5.2. We include only the indecomposable Lie algebras for simplicity. We note that the Lie bracket relations in these tables differ from the ones in the aforementioned papers by a minus sign, in the sense that we replace the basis element E_3 by $-E_3$ in Table 5.1, and E_4 by $-E_4$ in Table 5.2. This will reduce the number of minus signs later without affecting the validity of the classification.

Observe from Table 5.1 that for the Lie algebras $\mathfrak{g}_{3.1}, \dots, \mathfrak{g}_{3.5}^\alpha$, the basis elements E_1 and E_2 commute and hence form a copy of the abelian Lie algebra $\mathfrak{r}^2$. Furthermore, the adjoint mapping ad_{E_3} acts as a derivation on the Lie algebra spanned by E_1 and E_2 . Hence these Lie algebras are in fact semidirect products $\mathfrak{r} \ltimes \mathfrak{r}^2$. By the same token, a quick inspection of Table 5.2 reveals that the first six Lie algebras $\mathfrak{g}_{4.1}, \dots, \mathfrak{g}_{4.6}$ are semidirect products of the form $\mathfrak{r} \ltimes \mathfrak{r}^3$, the Lie algebras $\mathfrak{g}_{4.7}, \mathfrak{g}_{4.8}, \mathfrak{g}_{4.9}$ are semidirect products of the form $\mathfrak{r} \ltimes \mathfrak{h}^3$, where $\mathfrak{h}^3$ is the nilpotent three-dimensional Heisenberg algebra. Finally the Lie algebra $\mathfrak{g}_{4.10}$ is a semidirect product of the form $\mathfrak{r}^2 \ltimes \mathfrak{r}^2$. Now it is well-known that if G and H are connected simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, then the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ is the Lie algebra of a unique semidirect product of the corresponding Lie groups $G \ltimes H$. In particular, this implies that the simply connected four-dimensional Lie groups, whose Lie algebra is indecomposable, are semidirect products of the form $\mathbb{R} \ltimes \mathbb{R}^3$, $\mathbb{R}^2 \ltimes \mathbb{R}^2$, and $\mathbb{R} \ltimes \mathbb{H}^3$, where $\mathbb{H}^3$ denotes the three-dimensional Heisenberg group. These are indeed specific cases of the semidirect products that we will consider.

This motivates our interest in studying the more general Lie algebras which are semidirect products $\mathfrak{r}^m \ltimes_\pi \mathfrak{r}^n$ and $\mathfrak{r}^m \ltimes_\pi \mathfrak{h}^{2n+1}$, as well as the corresponding connected simply connected Lie groups $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$.

Lie algebra	Parameters	Lie brackets, basis $\{E_1, E_2, E_3\}$		
$\mathfrak{g}_{3.1} = \mathfrak{h}^3$		$[E_3, E_1] = 0,$	$[E_3, E_2] = E_1,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{3.2}$		$[E_3, E_1] = E_1,$	$[E_3, E_2] = E_1 + E_2,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{3.3}$		$[E_3, E_1] = E_1,$	$[E_3, E_2] = E_2,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{3.4}^\alpha$	$\alpha \in [-1, 1) \setminus \{0\}$	$[E_3, E_1] = E_1,$	$[E_3, E_2] = \alpha E_2,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{3.5}^\alpha$	$\alpha \geq 0$	$[E_3, E_1] = \alpha E_1 - E_2,$	$[E_3, E_2] = E_1 + \alpha E_2,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{3.6} = \mathfrak{sl}_2(\mathbb{R})$		$[E_3, E_1] = 2E_2,$	$[E_3, E_2] = E_3,$	$[E_2, E_1] = -E_1$
$\mathfrak{g}_{3.7} = \mathfrak{su}(2)$		$[E_3, E_1] = -E_2,$	$[E_3, E_2] = E_1,$	$[E_2, E_1] = -E_3$

Table 5.1: [37] The (indecomposable) three-dimensional Lie algebras

Lie algebra	Parameters	Lie brackets, basis $\{E_1, E_2, E_3, E_4\}$		
$\mathfrak{g}_{4.1}$		$[E_4, E_1] = 0,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = E_1,$ $[E_3, E_2] = 0,$	$[E_4, E_3] = E_2,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.2}^\alpha$	$\alpha \neq 0$	$[E_4, E_1] = \alpha E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = E_2,$ $[E_3, E_2] = 0,$	$[E_4, E_3] = E_2 + E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.3}$		$[E_4, E_1] = E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = 0,$ $[E_3, E_2] = 0,$	$[E_4, E_3] = E_2,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.4}$		$[E_4, E_1] = E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = E_1 + E_2,$ $[E_3, E_2] = 0,$	$[E_4, E_3] = E_2 + E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.5}^{\alpha\beta\gamma}$	$\alpha\beta\gamma \neq 0$	$[E_4, E_1] = \alpha E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = \beta E_2,$ $[E_3, E_2] = 0,$	$[E_4, E_3] = \gamma E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.6}^{\alpha\beta}$	$\alpha > 0,$ $\beta \in \mathbb{R}$	$[E_4, E_1] = \alpha E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = \beta E_2 - E_3,$ $[E_3, E_2] = 0,$	$[E_4, E_3] = E_2 + \beta E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.7}$		$[E_4, E_1] = 2E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = E_2,$ $[E_3, E_2] = -E_1,$	$[E_4, E_3] = E_2 + E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.8}^\alpha$	$\alpha \in [-1, 1]$	$[E_4, E_1] = (1 + \alpha)E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = E_2,$ $[E_3, E_2] = -E_1,$	$[E_4, E_3] = \alpha E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.9}^\alpha$	$\alpha \geq 0$	$[E_4, E_1] = 2\alpha E_1,$ $[E_3, E_1] = 0,$	$[E_4, E_2] = \alpha E_2 - E_3,$ $[E_3, E_2] = -E_1,$	$[E_4, E_3] = E_2 + \alpha E_3,$ $[E_2, E_1] = 0$
$\mathfrak{g}_{4.10}$		$[E_4, E_1] = -E_2,$ $[E_3, E_1] = -E_1,$	$[E_4, E_2] = E_1,$ $[E_3, E_2] = -E_2,$	$[E_4, E_3] = 0,$ $[E_2, E_1] = 0$

Table 5.2: [37] The (indecomposable) four-dimensional Lie algebras

5.2 Semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$

Throughout this section, m and n will be positive integers. Unless otherwise specified, the indices i, j are always assumed to take values between 1 and n , while the indices k, l take values between 1 and m .

We begin by considering Lie algebra semidirect products $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{r}^n$ with respect to a homomorphism $\pi : \mathfrak{r}^m \rightarrow \text{Der}(\mathfrak{r}^n) = \mathfrak{gl}_n(\mathbb{R})$. Observe that since $\mathfrak{r}^m$ is abelian and π is a homomorphism, the image algebra $\pi(\mathfrak{r}^m)$ is an abelian subalgebra $\mathfrak{gl}_n(\mathbb{R})$.

Let us find a linear representation of such semidirect products. By definition, the Lie brackets on the semidirect product $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{r}^n$ are given by

$$[T_k, T_l] = 0, \quad [X_i, X_j] = 0, \quad [T_k, X_j] = \pi(T_k)X_j = \sum_{i=1}^n (A_k)_{ij} X_i, \quad (5.1)$$

where $\{T_1, \dots, T_m\}$ and $\{X_1, \dots, X_n\}$ are abstract bases for $\mathfrak{r}^m$ and $\mathfrak{r}^n$, respectively, and the homomorphism π is defined by

$$\pi(T_k) = A_k$$

for some family $\mathcal{A} = (A_k)_{k=1}^m \subset \mathbb{R}^{n \times n}$ of matrices expressed in the basis $\{X_1, \dots, X_n\}$. Observe that the matrices A_k commute since the image $\pi(\mathfrak{r}^m)$ is abelian. Aiming to find a linear representation of T_k and X_i for which these relations hold, we consider the $(n+1+m) \times (n+1+m)$ matrices

$$T_k = \begin{bmatrix} A_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_k \end{bmatrix}, \quad X_i = \begin{bmatrix} 0_n & e_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_m \end{bmatrix},$$

where D_k are the $m \times m$ diagonal matrices with 1 on the k -th position of their diagonal and zeroes elsewhere, and $e_i \in \mathbb{R}^n$ are the canonical unit column vectors. Observe that the block involving the matrices D_k ensures that the matrices T_k are linearly independent, even if the matrices A_k are not. An elementary calculation shows that the commutator relations (5.1) indeed hold, so that the matrices T_k, X_i induce a faithful representation of the semidirect product $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{r}^n$.

Note that every Lie algebra of type $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{r}^n$ is solvable. Hence, the corresponding simply connected Lie group $\mathbb{R}^m \ltimes_{\mu} \mathbb{R}^n$, where $\mu : \mathbb{R}^m \rightarrow \text{Aut}(\mathbb{R}^n)$ is the action uniquely determined by π , will be diffeomorphic to $\mathbb{R}^m \times \mathbb{R}^n$. Aiming to find such a diffeomorphism, we consider the mapping

$$\begin{aligned} \mathbb{R}^m \times \mathbb{R}^n \ni (t, x) &\mapsto \text{Exp}(\sum_i X_i x_i) \cdot \text{Exp}(\sum_k T_k t_k) \\ &= \begin{bmatrix} \text{Exp}(\sum_k A_k t_k) & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{Exp}(\sum_k D_k t_k) \end{bmatrix}, \end{aligned} \quad (5.2)$$

which is indeed a diffeomorphism onto its image, since

$$\text{Exp}(\sum_k D_k t_k) = \text{Exp}(\text{diag}(t)) = \begin{bmatrix} e^{t_1} & 0 & \dots & 0 \\ 0 & e^{t_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{t_m} \end{bmatrix}. \quad (5.3)$$

A simple calculation confirms that the image of the mapping (5.2) is indeed the semidirect product $\mathbb{R}^m \ltimes_{\mu} \mathbb{R}^n$, where the action $\mu : \mathbb{R}^m \rightarrow \text{Aut}(\mathbb{R}^n)$ is given by

$$\mu(t) = \text{Exp}(\sum_k A_k t_k).$$

This means that in this representation we may view the semidirect product $\mathbb{R}^m \ltimes_{\mu} \mathbb{R}^n$ as the space $\mathbb{R}^m \times \mathbb{R}^n$ equipped with the group operation

$$(t, x)(s, y) = (t + s, x + \mu(t)y).$$

Note that the action μ is completely determined by the family $\mathcal{A} = (A_k)_{k=1}^m$, so it is natural to use the suggestive notation $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$ to denote such semidirect products.

Remark 5.2.1. We would like to note that the final m -dimensional block of the representation, i.e. the block (5.3), ensures that the mapping (5.2) is a diffeomorphism onto its image. This is needed in general since the mapping $t \mapsto \mu(t)$ may not even be injective for an arbitrary family $\mathcal{A}$. However, one can in many cases omit parts of this final m -dimensional block, or even omit the whole block, without affecting the validity of the representation.

In Tables 5.3 and 5.4, we list the linear representations of the three- and four-dimensional semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$ according to the classifications given in Tables 5.1 and 5.2. In view of Remark 5.2.1, we omit parts of the final m -dimensional block in the representation (5.2) whenever possible. Some of the representations that we present in these tables have already appeared in the papers [37, 9].

Lie group	Parameters	Family $\mathcal{A}$	Linear representation
$G_{3.1} = H^3$		$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & t & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix}$
$G_{3.2}$		$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^t & te^t & x_1 \\ 0 & e^t & x_2 \\ 0 & 0 & 1 \end{bmatrix}$
$G_{3.3}$		$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^t & 0 & x_1 \\ 0 & e^t & x_2 \\ 0 & 0 & 1 \end{bmatrix}$
$G_{3.4}^{\alpha}$	$\alpha \in [-1, 1) \setminus \{0\}$	$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$	$\begin{bmatrix} e^t & 0 & x_1 \\ 0 & e^{\alpha t} & x_2 \\ 0 & 0 & 1 \end{bmatrix}$
$G_{3.5}^{\alpha}$	$\alpha > 0$	$\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} \cos(t) & e^{\alpha t} \sin(t) & x_1 \\ -e^{\alpha t} \sin(t) & e^{\alpha t} \cos(t) & x_2 \\ 0 & 0 & 1 \end{bmatrix}$
	$\alpha = 0$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos(t) & \sin(t) & x_1 & 0 \\ -\sin(t) & \cos(t) & x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{bmatrix}$

Table 5.3: Linear representations of the semidirect products $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$ with $m+n = 3$

Lie group	Parameters	Family $\mathcal{A}$	Linear representation
$G_{4.1}$		$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & x_1 \\ 0 & 1 & t & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.2}^\alpha$	$\alpha \neq 0$	$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 & 0 & x_1 \\ 0 & e^t & te^t & x_2 \\ 0 & 0 & e^t & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.3}$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} e^t & 0 & 0 & x_1 \\ 0 & 1 & t & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.4}$		$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t & x_1 \\ 0 & e^t & te^t & x_2 \\ 0 & 0 & e^t & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.5}^{\alpha\beta\gamma}$	$\alpha\beta\gamma \neq 0$	$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 & 0 & x_1 \\ 0 & e^{\beta t} & 0 & x_2 \\ 0 & 0 & e^{\gamma t} & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.6}^{\alpha\beta}$	$\alpha > 0, \beta \in \mathbb{R}$	$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 1 \\ 0 & -1 & \beta \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 & 0 & x_1 \\ 0 & e^{\beta t} \cos(t) & e^{\beta t} \sin(t) & x_2 \\ 0 & -e^{\beta t} \sin(t) & e^{\beta t} \cos(t) & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.10}$		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} e^{-t_1} \cos(t_2) & e^{-t_1} \sin(t_2) & x_1 & 0 \\ -e^{-t_1} \sin(t_2) & e^{-t_1} \cos(t_2) & x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{t_2} \end{bmatrix}$

Table 5.4: Linear representations of the semidirect products $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$ with $m+n = 4$

We equip these semidirect products with the left-invariant Riemannian metric such that the basis matrix elements T_k, X_i of the Lie algebra $\mathfrak{r}^m \ltimes \mathfrak{r}^n$ becomes orthonormal. Let now us find an explicit formula for this metric on $\mathbb{R}^m \ltimes \mathbb{R}^n$ in the coordinates (t, x) . Let $p \in \mathbb{R}^m \ltimes \mathbb{R}^n$ correspond to the point $(t, x) \in \mathbb{R}^m \times \mathbb{R}^n$ in the coordinate system, so that

$$p = p(t, x) = \text{Exp}(\sum_i X_i x_i) \cdot \text{Exp}(\sum_k T_k t_k).$$

A direct calculation shows that

$$p^{-1} \frac{\partial}{\partial t_k} \Big|_p = T_k,$$

$$p^{-1} \frac{\partial}{\partial x_i} \Big|_p = \text{Exp}(-\sum_k T_k t_k) \cdot X_i \cdot \text{Exp}(\sum_k T_k t_k) = \sum_{j=1}^n \mu(-t)_{ji} X_j.$$

In view of the discussion at the end of Section 2.3, this implies that the canonical left-invariant metric on $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$ is given by

$$g_{(t,x)} = \begin{bmatrix} I_m & 0 \\ 0 & \mu(-t)^T \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & \mu(-t) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & \mu(-t)^T \mu(-t) \end{bmatrix}.$$

This leads to the following natural definition.

Definition 5.2.2. Let m, n be positive integers, let

$$\mathcal{A} = (A_k)_{k=1}^m \subset \mathbb{R}^{n \times n}$$

be an indexed family of commuting matrices and define $\mu : \mathbb{R}^m \rightarrow \mathbf{GL}_n(\mathbb{R})$ by

$$\mu(t) = \text{Exp} \left(\sum_{k=1}^m A_k t_k \right).$$

The *Riemannian semidirect product of $\mathbb{R}^m$ and $\mathbb{R}^n$ with respect to $\mathcal{A}$* is the solvable Lie group $\mathbb{R}^m \times \mathbb{R}^n$ with group operation

$$(t, x)(s, y) = (t + s, x + \mu(t)y),$$

equipped with the left-invariant Riemannian metric

$$g_{(t,x)} = \begin{bmatrix} I_m & 0 \\ 0 & \mu(-t)^T \mu(-t) \end{bmatrix}.$$

We denote such semidirect products by $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$.

We continue by finding an expression for the Laplace-Beltrami operator τ and the conformality operator κ in the coordinates (t, x) on $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$. We first observe that the cometric is given by

$$g_{(t,x)}^{-1} = \begin{bmatrix} I_m & 0 \\ 0 & \mu(t) \mu(t)^T \end{bmatrix},$$

and the determinant of g satisfies

$$\sqrt{\det g_{(t,x)}} = \sqrt{\det \mu(-t)^T \cdot \det \mu(-t)} = \exp \left(- \sum_k \text{Tr}(A_k) t_k \right).$$

If we let $\phi, \psi : \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^n \rightarrow \mathbb{C}$ be two complex-valued functions, we therefore see, from the local representation formulas (2.2), that the Laplace-Beltrami operator satisfies

$$\begin{aligned} \tau(\phi) &= \frac{1}{\sqrt{\det g}} \left\{ \sum_{k=1}^m \frac{\partial}{\partial t_k} \left(\sqrt{\det g} \frac{\partial \phi}{\partial t_k} \right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left((\mu(t) \mu(t)^T)_{ij} \sqrt{\det g} \frac{\partial \phi}{\partial x_j} \right) \right\} \\ &= \sum_{k=1}^m \left(\frac{\partial^2 \phi}{\partial t_k^2} - \text{Tr}(A_k) \frac{\partial \phi}{\partial t_k} \right) + \sum_{i,j=1}^n (\mu(t) \mu(t)^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \end{aligned} \quad (5.4)$$

while the conformality operator is given by

$$\kappa(\phi, \psi) = \sum_{k=1}^m \frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial t_k} + \sum_{i,j=1}^n (\mu(t) \mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j}, \quad (5.5)$$

where the mixed terms of the form $\frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial x_i}$ and $\frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial t_k}$ are not present since the corresponding components of g^{-1} are 0.

5.3 Semidirect products $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

Throughout this section $m, n \geq 1$ will be integers. In contrast to the previous section, the indices i, j here are assumed to take values between 1 and $2n$, while the indices k, l still take values between 1 and m . We denote by J_{2n} the block diagonal $2n \times 2n$ matrix

$$J_{2n} = \text{diag} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{n \text{ times}} \right) \in \mathbb{R}^{2n \times 2n},$$

so that if $x = (x_1, x_2, \dots, x_{2n-1}, x_{2n}) \in \mathbb{R}^{2n}$, then

$$J_{2n}x = (-x_2, x_1, \dots, -x_{2n}, x_{2n-1}).$$

Note that this is the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We begin by finding a suitable representation of the Heisenberg algebra $\mathfrak{h}^{2n+1}$ and its corresponding simply connected Heisenberg group $\mathbb{H}^{2n+1}$. For this we consider the $(1 + 2n + 1 + m) \times (1 + 2n + 1 + m)$ matrices

$$\Xi = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0_{2n} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_m \end{bmatrix}, \quad X_i = \begin{bmatrix} 0 & \frac{1}{2}(J_{2n}e_i)^T & 0 & 0 \\ 0 & 0_{2n} & e_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_m \end{bmatrix},$$

where e_i are the canonical unit column vectors of $\mathbb{R}^{2n}$. A simple calculation shows that for $1 \leq i \leq j \leq 2n$,

$$[\Xi, X_i] = 0, \quad [X_i, X_j] = \begin{cases} \Xi, & i \text{ odd and } j = i + 1 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, these matrices induce a faithful representation of the nilpotent $(2n + 1)$ -dimensional Heisenberg algebra $\mathfrak{h}^{2n+1}$ as a subalgebra of $\mathfrak{gl}_{1+2n+1+m}(\mathbb{R})$. Note that the final m -dimensional block is superfluous at the moment, but it will be needed for the representation of semidirect products $\mathfrak{r}^m \ltimes \mathfrak{h}^{2n+1}$. Since this Lie algebra is nilpotent, the exponential mapping provides us with a diffeomorphism onto the simply connected Heisenberg group $\mathbb{H}^{2n+1}$ i.e. we obtain the representation

$$\mathbb{H}^{2n+1} = \left\{ \text{Exp} \left(\sum_i X_i x_i + \Xi \xi \right) = \begin{bmatrix} 1 & \frac{1}{2}(J_{2n}x)^T & \xi & 0 \\ 0 & I_{2n} & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \mid (\xi, x) \in \mathbb{R} \times \mathbb{R}^{2n} \right\}.$$

In this representation, we can view $\mathbb{H}^{2n+1}$ as the space $\mathbb{R} \times \mathbb{R}^{2n}$ equipped with the operation

$$(\xi, x) \boxplus (\eta, y) = \left(\xi + \eta + \frac{1}{2} \langle J_{2n}x, y \rangle, x + y \right). \quad (5.6)$$

We now find a representation of semidirect products $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{h}^{2n+1}$ with respect to a Lie algebra homomorphism $\pi : \mathfrak{r}^m \rightarrow \text{Der}(\mathfrak{h}^{2n+1})$. We again observe that since $\mathfrak{r}^m$ is abelian, so is the image $\pi(\mathfrak{r}^m) \subset \text{Der}(\mathfrak{h}^{2n+1})$ since π is a homomorphism. An easy calculation, which can be found in Proposition 9 of [14], shows that the algebra

$\text{Der}(\mathfrak{h}^{2n+1})$ of derivations on $\mathfrak{h}^{2n+1}$ consists of endomorphisms, represented in the basis $\{\Xi, X_1, \dots, X_{2n}\}$, of the form

$$\begin{bmatrix} a & v^T \\ 0 & A \end{bmatrix} \in \mathbb{R}^{(1+2n) \times (1+2n)}, \quad (5.7)$$

where $a \in \mathbb{R}$ and the column vector $v \in \mathbb{R}^{2n}$ are arbitrary, while the matrix $A \in \mathbb{R}^{2n \times 2n}$ is of the block form

$$A = \begin{bmatrix} A_{(1,1)} & -\text{adj}(A_{(2,1)}) & -\text{adj}(A_{(3,1)}) & \dots & -\text{adj}(A_{(n,1)}) \\ A_{(2,1)} & A_{(2,2)} & -\text{adj}(A_{(3,2)}) & \dots & -\text{adj}(A_{(n,2)}) \\ A_{(3,1)} & A_{(3,2)} & A_{(3,3)} & \dots & -\text{adj}(A_{(n,3)}) \\ \vdots & & & \ddots & \vdots \\ A_{(n,1)} & A_{(n,2)} & A_{(n,3)} & \dots & A_{(n,n)} \end{bmatrix}, \quad (5.8)$$

where $A_{(i,j)} \in \mathbb{R}^{2 \times 2}$ are such that $\text{Tr } A_{(i,i)} = a$ for $1 \leq i \leq n$, and $\text{adj}(A_{(i,j)})$ denotes the adjugate i.e. the transpose of the cofactor matrix of $A_{(i,j)}$. In what follows we will only consider homomorphisms such that any element of the image satisfies $v = 0$, cf. (5.7). In view of the classification in Table 5.2, this can be assumed for all of the four-dimensional semidirect products $\mathfrak{r} \ltimes \mathfrak{h}^3$, as we will now see. Note that we can then represent every element of the image $\pi(\mathfrak{r}^m)$ as

$$\begin{bmatrix} \frac{1}{n} \text{Tr}(A) & 0 \\ 0 & A \end{bmatrix},$$

where the matrix A is of the form (5.8), and in particular we see that each such element is completely determined by the matrix A . By definition, the non-zero Lie brackets on the semidirect product $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{h}^{2n+1}$ must satisfy

$$\begin{aligned} [X_i, X_j] &= \begin{cases} \Xi, & i \text{ odd and } j = i + 1 \\ 0, & \text{otherwise} \end{cases} \\ [T_k, \Xi] &= \pi(T_k) \Xi = \frac{1}{n} \text{Tr}(A_k) \Xi, \\ [T_k, X_j] &= \pi(T_k) X_j = \sum_i (A_k)_{ij} X_i. \end{aligned} \quad (5.9)$$

Here $\{T_1, \dots, T_m\}$ and $\{\Xi, X_1, \dots, X_{2n}\}$ are abstract bases for $\mathfrak{r}^m$ and $\mathfrak{h}^{2n+1}$, respectively, and the homomorphism π is defined by

$$\pi(T_k) = \begin{bmatrix} \frac{1}{n} \text{Tr}(A_k) & 0 \\ 0 & A_k \end{bmatrix}$$

for some matrices $A_k \in \mathbb{R}^{2n \times 2n}$ of the form (5.8) which commute, since $\pi(\mathfrak{r}^m)$ is abelian. We have already found a suitable representation for the Heisenberg algebra spanned by $\Xi, X_1, \dots, X_{2n}$, so to complete the representation we further consider the $(1 + 2n + 1 + m) \times (1 + 2n + 1 + m)$ matrices

$$T_k = \begin{bmatrix} \frac{1}{n} \text{Tr}(A_k) & 0 & 0 & 0 \\ 0 & A_k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_k \end{bmatrix},$$

where D_k are the $m \times m$ diagonal matrices with 1 on the k -th position of their diagonal and zeroes elsewhere. Then an easy calculation shows that the relations (5.9) hold, so that the matrices T_k, Ξ, X_i induce a faithful representation of the semidirect product $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{h}^{2n+1}$.

Just as in the previous section, since the Lie algebra $\mathfrak{r}^m \ltimes_{\pi} \mathfrak{h}^{2n+1}$ is solvable, the corresponding simply connected Lie group is diffeomorphic to $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n}$. In order to find this diffeomorphism, we consider the mapping

$$\begin{aligned} \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n} \ni (t, \xi, x) &\mapsto \text{Exp}(\sum_i X_i x_i + \Xi \xi) \cdot \text{Exp}(\sum_k T_k t_k) \\ &= \begin{bmatrix} a(t) & \frac{1}{2}(J_{2n}x)^T \mu(t) & \xi & 0 \\ 0 & \mu(t) & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \text{Exp}(\text{diag}(t)) \end{bmatrix}, \end{aligned}$$

where

$$a(t) = \exp\left(\frac{1}{n} \sum_{k=1}^m \text{Tr}(A_k) t_k\right) \quad \text{and} \quad \mu(t) = \text{Exp}\left(\sum_{k=1}^m A_k t_k\right).$$

This is clearly a diffeomorphism onto its image, and one can show that its image is indeed the semidirect product $\mathbb{R}^m \ltimes_{\hat{\mu}} \mathbb{H}^{2n+1}$ with respect to the action $\hat{\mu} : \mathbb{R}^m \rightarrow \text{Aut}(\mathbb{H}^{2n+1})$ given by

$$\hat{\mu}(t) = \begin{bmatrix} a(t) & 0 \\ 0 & \mu(t) \end{bmatrix}.$$

We omit the details of this calculation, but let us at least note that one is required to use the identity

$$J_{2n}^T \mu(t) = a(t) (J_{2n} \mu(-t))^T. \quad (5.10)$$

The proof of this identity is not difficult but requires some work, so we include it in Appendix A.2. In this representation, we can thus view $\mathbb{R}^m \ltimes_{\hat{\mu}} \mathbb{H}^{2n+1}$ as the space $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n+1}$ equipped with the Lie group operation

$$\begin{aligned} (t, \xi, x)(s, \eta, y) &= (t + s, (\xi, x) \boxplus (a(t)\eta, \mu(t)y)) \\ &= (t + s, \xi + a(t)\eta + \frac{1}{2}\langle J_{2n}x, \mu(t)y \rangle, x + \mu(t)y), \end{aligned}$$

where $\boxplus$ is the group operation on the Heisenberg group defined in (5.6).

In Table 5.5, we list the linear representations of semidirect products $\mathbb{R} \ltimes \mathbb{H}^3$ according to the classification in Table 5.2. As in the case of $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$, one can often omit parts of the final m -dimensional block without affecting the validity of the representation, cf. Remark 5.2.1. In Table 5.5 we do so whenever possible.

We equip these semidirect products with the left-invariant Riemannian metric such that the basis matrix T_i, Ξ, X_k of the Lie algebra $\mathfrak{r}^m \ltimes \mathfrak{h}^{2n+1}$ becomes orthonormal. To find an explicit formula for this metric in the coordinates (t, ξ, x) , let $p = p(t, \xi, x) \in \mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$ correspond to the point $(t, \xi, x) \in \mathbb{R}^m \times \mathbb{H}^{2n+1}$, so that

$$p = \text{Exp}(\sum_i X_i x_i + \Xi \xi) \cdot \text{Exp}(\sum_k T_k t_k).$$

Lie group	Parameters	Family $\mathcal{A}$	Linear representation
$G_{4.7}$		$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^{2t} & -\frac{x_2}{2}e^t & \frac{x_1-tx_2}{2}e^t & \xi \\ 0 & e^t & te^t & x_1 \\ 0 & 0 & e^t & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.8}^\alpha$	$\alpha \in [-1, 1]$	$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$	$\begin{bmatrix} e^{(1+\alpha)t} & -\frac{x_2}{2}e^t & \frac{x_1}{2}e^{\alpha t} & \xi \\ 0 & e^t & 0 & x_1 \\ 0 & 0 & e^{\alpha t} & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4.9}^\alpha$	$\alpha > 0$	$\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$	$\begin{bmatrix} e^{2\alpha t} & -\frac{x_2 \cos(t)+x_1 \sin(t)}{2}e^{\alpha t} & \frac{x_1 \cos(t)-x_2 \sin(t)}{2}e^{\alpha t} & \xi \\ 0 & e^{\alpha t} \cos(t) & e^{\alpha t} \sin(t) & x_1 \\ 0 & -e^{\alpha t} \sin(t) & e^{\alpha t} \cos(t) & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
	$\alpha = 0$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -\frac{x_2 \cos(t)+x_1 \sin(t)}{2} & \frac{x_1 \cos(t)-x_2 \sin(t)}{2} & \xi & 0 \\ 0 & \cos(t) & \sin(t) & x_1 & 0 \\ 0 & -\sin(t) & \cos(t) & x_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^t \end{bmatrix}$

Table 5.5: Linear representations of the semidirect products $\mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3$

A simple calculation shows that

$$\begin{aligned}
p^{-1} \frac{\partial}{\partial t_k} \Big|_p &= T_k, \\
p^{-1} \frac{\partial}{\partial \xi} \Big|_p &= \text{Exp}(-\sum_k T_k t_k) \cdot \Xi \cdot \text{Exp}(\sum_k T_k t_k), \\
p^{-1} \frac{\partial}{\partial x_i} \Big|_p &= \text{Exp}(-\sum_k T_k t_k) \cdot (X_i - \frac{1}{2}(J_{2n}x)_i \Xi) \cdot \text{Exp}(\sum_k T_k t_k),
\end{aligned}$$

and furthermore,

$$\begin{aligned}
&\text{Exp}(-\sum_k T_k t_k) \cdot \Xi \cdot \text{Exp}(\sum_k T_k t_k) = a(-t) \Xi, \\
&\text{Exp}(-\sum_k T_k t_k) \cdot X_i \cdot \text{Exp}(\sum_k T_k t_k) = \sum_{j=1}^{2n} \mu(-t)_{ji} X_j.
\end{aligned}$$

To prove the latter, one must again employ the identity (5.10). In view of the discussion at the end of Section 2.3, we see that the canonical left-invariant Riemannian metric in this parametrization is given by

$$\begin{aligned}
g_{(t,\xi,x)} &= \begin{bmatrix} I_m & 0 & 0 \\ 0 & a(-t) & 0 \\ 0 & -\frac{1}{2}a(-t)J_{2n}x & \mu(-t)^T \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & a(-t) & -\frac{1}{2}a(-t)(J_{2n}x)^T \\ 0 & 0 & \mu(-t) \end{bmatrix} \\
&= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2}J_{2n}x & I_{2n} \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & a(-t)^2 & 0 \\ 0 & 0 & \mu(-t)^T \mu(-t) \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & -\frac{1}{2}(J_{2n}x)^T \\ 0 & 0 & I_{2n} \end{bmatrix} \quad (5.11) \\
&= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu(-t)^T \mu(-t) \end{bmatrix} + a(-t)^2 \begin{bmatrix} 0_m & 0 & 0 \\ 0 & 1 & -\frac{1}{2}(J_{2n}x)^T \\ 0 & -\frac{1}{2}J_{2n}x & \frac{1}{4}J_{2n}x(J_{2n}x)^T \end{bmatrix}.
\end{aligned}$$

This leads to the following natural definition.

Definition 5.3.1. Let m, n be positive integers and

$$\mathcal{A} = (A_k)_{k=1}^m \subset \mathbb{R}^{2n \times 2n}$$

be an indexed family of commuting matrices of the form (5.8). Define $a : \mathbb{R}^m \rightarrow \mathbb{R}$ and $\mu : \mathbb{R}^m \rightarrow \mathbf{GL}_{2n}(\mathbb{R})$ by

$$a(t) = \exp \left(\frac{1}{n} \sum_{k=1}^m \text{Tr}(A_k) t_k \right) \quad \text{and} \quad \mu(t) = \text{Exp} \left(\sum_{k=1}^m A_k t_k \right).$$

The *Riemannian semidirect product* of $\mathbb{R}^m$ and the Heisenberg group $\mathbf{H}^{2n+1}$ with respect to $\mathcal{A}$ is the solvable Lie group $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n}$ with group operation

$$(t, \xi, x)(s, \eta, y) = (t + s, \xi + a(t)\eta + \frac{1}{2}\langle J_{2n}x, \mu(t)y \rangle, x + \mu(t)y),$$

equipped with the left-invariant Riemannian metric

$$g_{(t, \xi, x)} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu(-t)^T \mu(-t) \end{bmatrix} + a(-t)^2 \begin{bmatrix} 0_m & 0 & 0 \\ 0 & 1 & -\frac{1}{2}(J_{2n}x)^T \\ 0 & -\frac{1}{2}J_{2n}x & \frac{1}{4}J_{2n}x(J_{2n}x)^T \end{bmatrix}.$$

We will denote such semidirect products by $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbf{H}^{2n+1}$.

Let us now find an expression for the Laplace-Beltrami operator τ and the conformality operator κ on $\mathbb{R}^m \ltimes \mathbf{H}^{2n+1}$ in the coordinates (t, ξ, x) . Using the formula (5.11) for the metric on $\mathbb{R}^m \ltimes \mathbf{H}^{2n+1}$, we can easily compute the cometric

$$\begin{aligned} g_{(t, \xi, x)}^{-1} &= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & \frac{1}{2}(J_{2n}x)^T \\ 0 & 0 & I_{2n} \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & a(t)^2 & 0 \\ 0 & 0 & \mu(t)\mu(t)^T \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2}J_{2n}x & I_{2n} \end{bmatrix} \\ &= \begin{bmatrix} I_m & 0 & 0 \\ 0 & a(t)^2 + \frac{1}{4}\langle \mu(t)^T J_{2n}x, \mu(t)^T J_{2n}x \rangle & \frac{1}{2}(J_{2n}x)^T \mu(t)\mu(t)^T \\ 0 & \frac{1}{2}\mu(t)\mu(t)^T J_{2n}x & \mu(t)\mu(t)^T \end{bmatrix}, \end{aligned}$$

and we also see that the determinant satisfies

$$\sqrt{\det g_{(t, \xi, x)}} = a(-t) \det \mu(-t) = \exp \left(-\frac{n+1}{n} \sum_k \text{Tr}(A_k) t_k \right).$$

In particular, for two complex-valued functions $\phi, \psi : \mathbb{R}^m \ltimes_{\mathcal{A}} \mathbf{H}^{2n+1} \rightarrow \mathbb{C}$ we immediately get from the local representation formula (2.2) that the conformality operator satisfies

$$\begin{aligned} \kappa(\phi, \psi) &= \sum_{k=1}^m \frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial t_k} + \left(a(t)^2 + \frac{1}{4}\langle \mu(t)^T J_{2n}x, \mu(t)^T J_{2n}x \rangle \right) \frac{\partial \phi}{\partial \xi} \frac{\partial \psi}{\partial \xi} \\ &\quad + \frac{1}{2} \sum_{i=1}^{2n} (\mu(t)\mu(t)^T J_{2n}x)_i \left(\frac{\partial \phi}{\partial \xi} \frac{\partial \psi}{\partial x_i} + \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial \xi} \right) \\ &\quad + \sum_{i,j=1}^{2n} (\mu(t)\mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j}. \end{aligned} \tag{5.12}$$

To obtain an expression for the Laplace-Beltrami operator, we first note that since $\det g$ depends only on t , it will only affect the derivatives with respect to the t -variables in formula (2.2). Hence,

$$\begin{aligned}
\tau(\phi) &= \frac{1}{\sqrt{\det g}} \sum_{k=1}^m \frac{\partial}{\partial t_k} \left(\sqrt{\det g} \frac{\partial \phi}{\partial t_k} \right) + \frac{\partial}{\partial \xi} \left(\left(a(t)^2 + \frac{1}{4} \langle \mu(t)^T J_{2n} x, \mu(t)^T J_{2n} x \rangle \right) \frac{\partial \phi}{\partial \xi} \right) \\
&\quad + \sum_{i=1}^{2n} \frac{\partial}{\partial \xi} \left(\frac{\partial \phi}{\partial x_i} \left(\frac{1}{2} (J_{2n} x)^T \mu(t) \mu(t)^T \right)_i \right) + \sum_{i=1}^{2n} \frac{\partial}{\partial x_i} \left(\left(\frac{1}{2} \mu(t) \mu(t)^T J_{2n} x \right)_i \frac{\partial \phi}{\partial \xi} \right) \\
&\quad + \sum_{i,j=1}^{2n} \frac{\partial}{\partial x_i} \left((\mu(t) \mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_j} \right) \\
&= \sum_{k=1}^m \left(\frac{\partial^2 \phi}{\partial t_k^2} - \frac{n+1}{n} \operatorname{Tr}(A_k) \frac{\partial \phi}{\partial t_k} \right) + \left(a(t)^2 + \frac{1}{4} \langle \mu(t)^T J_{2n} x, \mu(t)^T J_{2n} x \rangle \right) \frac{\partial^2 \phi}{\partial \xi^2} \\
&\quad + \sum_{i=1}^{2n} (\mu(t) \mu(t)^T J_{2n} x)_i \frac{\partial^2 \phi}{\partial \xi \partial x_i} + \sum_{i,j=1}^{2n} (\mu(t) \mu(t)^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{5.13}
\end{aligned}$$

Here we use the fact that

$$\begin{aligned}
&\sum_{i=1}^{2n} \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial \xi} \left(\frac{1}{2} \mu(t) \mu(t)^T J_{2n} x \right)_i \right) \\
&= \frac{1}{2} \sum_{i=1}^{2n} (\mu(t) \mu(t)^T J_{2n} x)_i \frac{\partial^2 \phi}{\partial x_i \partial \xi} + \frac{1}{2} \sum_{i=1}^{2n} (\mu(t) \mu(t)^T J_{2n} e_i)_i \frac{\partial \phi}{\partial \xi} \\
&= \frac{1}{2} \sum_{i=1}^{2n} (\mu(t) \mu(t)^T J_{2n} x)_i \frac{\partial^2 \phi}{\partial x_i \partial \xi} + \frac{1}{2} \operatorname{Tr}(\mu(t) \mu(t)^T J_{2n}) \frac{\partial \phi}{\partial \xi},
\end{aligned}$$

and the last term vanishes because the product of a symmetric and a skew-symmetric matrix is traceless.

5.4 p -Harmonic functions on $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

In this section we construct proper p -harmonic functions on the semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$. There are a lot of similarities between the constructions on these Lie groups that we will present, so we will study them in parallel. Aside from the notation used in Definitions 5.2.2 and 5.3.1, we also introduce the following notation, in order to simplify the statements of our results.

Notation 5.4.1. We will use the letter G to denote either $\mathbb{R}^n$ or $\mathbb{H}^{2n+1}$. If $G = \mathbb{R}^n$, the indices i, j will be assumed to take values between 1 and n , and if $G = \mathbb{H}^{2n+1}$, the indices i, j will be assumed to take values between 1 and $2n$, and in both cases the index k will be assumed to take values between 1 and m , unless otherwise specified. Throughout this section, $\mathcal{A} = (A_k)_k$ will denote a commuting family of real matrices, which are of dimensions $n \times n$ if $G = \mathbb{R}^n$, and of dimensions $2n \times 2n$ if $G = \mathbb{H}^{2n+1}$.

In the latter case, we also assume that each member of the family $\mathcal{A}$ is of the form (5.8). Finally, we will use the notation

$$\mathbb{R}^m \ni \omega = \begin{cases} (\text{Tr}(A_1), \dots, \text{Tr}(A_m)), & \text{if } G = \mathbb{R}^n \\ \frac{n+1}{n} (\text{Tr}(A_1), \dots, \text{Tr}(A_m)), & \text{if } G = \mathbb{H}^{2n+1} \end{cases}. \quad (5.14)$$

We note that with this notation, if $\phi, \psi : \mathbb{R}^m \times_{\mathcal{A}} G \rightarrow \mathbb{C}$ depend only on the coordinates t and x , i.e. if they are independent of ξ in the case $G = \mathbb{H}^{2n+1}$, then the Laplace-Beltrami operator satisfies

$$\tau(\phi) = \sum_k \left(\frac{\partial^2 \phi}{\partial t_k^2} - \omega_k \frac{\partial \phi}{\partial t_k} \right) + \sum_{ij} (\mu(t) \mu(t)^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j},$$

and the conformality operator is given by

$$\kappa(\phi, \psi) = \sum_k \frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial t_k} + \sum_{ij} (\mu(t) \mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j},$$

regardless of the choice of G .

We start with a general variable separation result.

Proposition 5.4.2. *Let $\phi, \psi : \mathbb{R}^m \times_{\mathcal{A}} G \rightarrow \mathbb{C}$ be two complex-valued functions such that ϕ depends only on $t \in \mathbb{R}^m$, while $\tau^\alpha(\psi)$ is independent of t for all $\alpha \geq 0$. Then their product $\phi\psi$ satisfies the identity*

$$\tau^p(\phi\psi) = \sum_{\alpha=0}^p \binom{p}{\alpha} \tau^\alpha(\phi) \tau^{p-\alpha}(\psi).$$

In particular, if ϕ and ψ are proper p -harmonic and proper r -harmonic on $\mathbb{R}^m \times_{\mathcal{A}} G$, respectively, then their product $\phi\psi$ is proper $(p+r-1)$ -harmonic on $\mathbb{R}^m \times_{\mathcal{A}} G$.

Remark 5.4.3. This result is reminiscent of the variable separation theorem on product manifolds, which can be found e.g. in Lemma 6.1 of [19]. For our result here we need the additional assumption that $\tau^\alpha(\psi)$ is independent of t for all $\alpha \geq 0$. This assumption is superfluous for direct products of manifolds, but it is essential in our case.

Proof. Since ϕ is a function of t , we see by the formulas (5.4) and (5.13) for the Laplace-Beltrami operator that $\tau^\beta(\phi)$ remains to be a function of t for all $\beta \geq 0$. Since the tension field $\tau^\alpha(\psi)$ is independent of t for all $\alpha \geq 0$ by assumption, it follows from the formulas (5.5) and (5.12) for the conformality operator that

$$\kappa(\tau^\beta(\phi), \tau^\alpha(\psi)) = 0,$$

for all $\alpha, \beta \geq 0$. The first part of the proposition now follows easily by induction combined with the product rule for the Laplace-Beltrami operator. For the final statement of the result, notice that the identity we have just proven implies that

$$\tau^{p+r-2}(\phi\psi) = \binom{p+r-2}{p-1} \tau^{p-1}(\phi) \tau^{r-1}(\psi) \neq 0 \quad \text{and} \quad \tau^{p+r-1}(\phi\psi) = 0,$$

so that $\phi\psi$ is indeed proper $(p+r-1)$ -harmonic. $\square$

Constructing proper p -harmonic functions $\psi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ such that $\tau^\alpha(\psi)$ is independent of t for $\alpha \geq 0$ seems difficult, if not impossible, in general. However, we will be able to do so later given that the elements of the family $\mathcal{A}$ possess a certain property. For now, we note that any harmonic function on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ which is independent of t trivially satisfies this condition. Such functions can easily be constructed as follows.

Proposition 5.4.4. *Define the complex-valued function $\psi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by*

$$\begin{cases} \psi(x) = a + \sum_i v_i x_i + \sum_{ij} B_{ij} x_i x_j, & \text{if } G = \mathbb{R}^n \\ \psi(\xi, x) = a + b\xi + \sum_i v_i x_i + \sum_{ij} B_{ij} x_i x_j, & \text{if } G = \mathbb{H}^{2n+1} \end{cases},$$

where $a, b, v_i, B_{ij} \in \mathbb{C}$ are not all zero, and the coefficients B_{ij} form a symmetric matrix such that

$$\text{Tr}(\mu(t) \mu(t)^T B) = 0, \quad t \in \mathbb{R}^m.$$

Then ψ is proper harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$.

Proof. Since B is symmetric, we obtain $\frac{\partial^2 \psi}{\partial x_i \partial x_j} = 2B_{ij}$, so that

$$\tau(\psi) = 2 \sum_{ij} (\mu(t) \mu(t)^T)_{ij} B_{ij} = 2 \text{Tr}(\mu(t) \mu(t)^T B),$$

confirming the result in both cases. $\square$

We can of course choose $B = 0_n$, but in some cases B can even be taken to be non-zero, as we will see in Examples 5.4.8 and 5.4.9 later. We continue by studying functions which depend only on t . In this case we can separate the variables even further.

Proposition 5.4.5. *For each $k = 1, \dots, m$, let $\phi_k : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ be a complex-valued function depending only on t_k . Then the product $\prod_k \phi_k$ satisfies the identity*

$$\tau^p \left(\prod_{k=1}^m \phi_k \right) = \sum_{j_1 + \dots + j_m = p} \binom{p}{j_1, \dots, j_m} \prod_{k=1}^m \tau^{j_k}(\phi_k).$$

In particular, if the function ϕ_k is proper p_k -harmonic for each k , then the product $\prod_k \phi_k$ is proper $(p_1 + \dots + p_m - m + 1)$ -harmonic.

Remark 5.4.6. Here we use the multinomial coefficients defined by

$$\binom{p}{j_1, \dots, j_m} = \frac{p!}{j_1! \cdots j_m!}.$$

Proof. The method of proof is similar to that of Proposition 5.4.2, the only difference being that one must also perform induction on m . $\square$

To make Proposition 5.4.5 useful, let us construct proper p -harmonic functions depending only on t_k . We observe that the coordinate functions t_k are isoparametric on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ in the sense of Definition 3.2.1, unlocking Theorem 3.2.2.

Proposition 5.4.7. For $k = 1, \dots, m$ and $p \geq 1$, define the complex-valued function $\phi_{k,p} : \mathbb{R}^m \times_{\mathcal{A}} G \rightarrow \mathbb{C}$ by

$$\phi_{k,p}(t_k) = \begin{cases} c_1 t_k^{p-1} e^{\omega_k t_k} + c_2 t_k^{p-1}, & \text{if } \omega_k \neq 0 \\ c_1 t_k^{2p-1} + c_2 t_k^{2p-2}, & \text{if } \omega_k = 0 \end{cases},$$

where $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$. Then $\phi_{k,p}$ is proper p -harmonic on $\mathbb{R}^m \times_{\mathcal{A}} G$.

Proof. Throughout the proof we use the same notation as in Theorem 3.2.2. A simple calculation gives

$$\tau(t_k) = -\omega_k \quad \text{and} \quad \kappa(t_k, t_k) = 1,$$

so that the coordinate function t_k is isoparametric, and its corresponding holomorphic functions $\Phi, \Psi : \mathbb{C} \rightarrow \mathbb{C}$ are given by

$$\Phi(z) = -\omega_k \quad \text{and} \quad \Psi(z) = 1.$$

It follows that

$$\Lambda(z) = e^{\omega_k z},$$

so that the function $f_1 : \mathbb{R} \rightarrow \mathbb{C}$ satisfies

$$f_1(z) = c_1 \int^z e^{\omega_k \zeta} d\zeta + c_2 = \begin{cases} \frac{c_1}{\omega_k} e^{\omega_k z} + c_2, & \text{if } \omega_k \neq 0 \\ c_1 z + c_2, & \text{if } \omega_k = 0 \end{cases},$$

which proves the result for $p = 1$. We proceed by induction. If $\omega_k = 0$ and f_p has the required form, we directly get

$$f_{p+1}(z) = \int^z \int^\eta (c_1 \zeta^{2p-1} + c_2 \zeta^{2p-2}) d\zeta d\eta = \frac{c_1}{2p(2p+1)} z^{2p+1} + \frac{c_2}{2p(2p-1)} z^{2p},$$

and the result follows. On the other hand, if $\omega_k \neq 0$, we use the fact that for any $\nu \in \mathbb{C} \setminus \{0\}$ and any non-negative integer n ,

$$\int^z \zeta^n e^{\nu \zeta} d\zeta = e^{\nu z} \sum_{j=0}^n a_j z^j,$$

for an appropriate choice of the constants $a_j \in \mathbb{C}$. Assuming that the functions $f_1, \dots, f_p$ have been shown to have the required form, we thus get

$$\begin{aligned} f_{p+1}(z) &= \int^z e^{\omega_k \eta} \int^\eta e^{-\omega_k \zeta} (c_1 \zeta^{p-1} e^{\omega_k \zeta} + c_2 \zeta^{p-1}) d\zeta d\eta \\ &= \frac{c_1}{p-1} \int^z \eta^p e^{\omega_k \eta} d\eta + c_2 \sum_{\alpha=0}^{p-1} a_\alpha \int^z \eta^\alpha d\eta \\ &= \frac{c_1}{p-1} \sum_{\beta=0}^p c_\beta z^\beta e^{\omega_k z} + c_2 \sum_{\alpha=0}^{p-1} \frac{a_\alpha}{\alpha} z^{\alpha+1}. \end{aligned}$$

The terms with $\beta < p$ and $\alpha < p-1$ will become p -harmonic after pulling back by t_k by the induction hypothesis, so that we may discard them for simplicity. After modifying the constants we get

$$f_{p+1}(z) = c_1 z^p e^{\omega_k z} + c_2 z^p,$$

which completes the proof. $\square$

Example 5.4.8 ($G_{4.8}^\alpha$). Consider the Lie group $G_{4.8}^\alpha = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3$, whose family $\mathcal{A}$ consists of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \quad \text{so that} \quad \mu(t) \mu(t)^\top = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2\alpha t} \end{bmatrix}.$$

We set

$$B = \frac{\lambda}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}$, so that the function ψ from Proposition 5.4.4 is given by

$$\psi(\xi, x) = a + b\xi + v_1x_1 + v_2x_2 + \lambda x_1x_2.$$

Then B evidently satisfies the condition from Proposition 5.4.4. Hence Proposition 5.4.2, in conjunction with Propositions 5.4.7 and 5.4.4, implies that the function defined by

$$(t, \xi, x) \mapsto \begin{cases} (c_1t^{2p-1} + c_2t^{2p-2})(a + b\xi + v_1x_1 + v_2x_2 + \lambda x_1x_2), & \text{if } \alpha = -1 \\ (c_1t^{p-1}e^{2(1+\alpha)t} + c_2t^{p-1})(a + b\xi + v_1x_1 + v_2x_2 + \lambda x_1x_2), & \text{otherwise} \end{cases}$$

is proper p -harmonic on $G_{4.8}^\alpha$. $\triangle$

Example 5.4.9 ($G_{4.4}$). Consider the Lie group $G_{4.4} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$, whose corresponding family $\mathcal{A}$ consists of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so that} \quad \mu(t) \mu(t)^\top = e^{-2t} \begin{bmatrix} 1 + t^2 + \frac{t^4}{4} & t + \frac{t^3}{2} & \frac{t^2}{2} \\ +t + \frac{t^3}{2} & 1 + t^2 & t \\ \frac{t^2}{2} & t & 1 \end{bmatrix}.$$

We set

$$B = \lambda \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix},$$

where $\lambda \in \mathbb{C}$, so that the function ψ from Proposition 5.4.4 is given by

$$\psi(x) = a + v_1x_1 + v_2x_2 + \lambda(x_2^2 - x_3^3 - 2x_1x_3).$$

It is easy to check that the condition from Proposition 5.4.4 is satisfied, so that Proposition 5.4.2, combined with Propositions 5.4.7 and 5.4.4, shows that the function defined by

$$(t, x) \mapsto (c_1t^{p-1}e^{-3t} + c_2t^{p-1})(a + v_1x_1 + v_2x_2 + \lambda(x_2^2 - x_3^3 - 2x_1x_3))$$

is proper p -harmonic on $G_{4.4}$. $\triangle$

We now proceed by applying Theorem 3.2.2 to a certain class of isoparametric functions on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$. These functions will depend only on the variables t and x i.e. they will be independent of the variable ξ if $G = \mathbb{H}^{2n+1}$. To construct this class of isoparametric functions, we will use the well-known fact that the elements of a commuting family of matrices always possess a common eigenvector.

Lemma 5.4.10. *Let v be a common eigenvector of the commuting family $\mathcal{A}^\mathsf{T} = (A_k^\mathsf{T})_k$, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the vector consisting of the corresponding eigenvalues. Then the function $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ defined by*

$$\phi(t, x) = e^{-\langle \lambda, t \rangle} \langle v, x \rangle$$

is isoparametric on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ with

$$\tau(\phi) = \langle \lambda, \lambda + \omega \rangle \phi \quad \text{and} \quad \kappa(\phi, \phi) = \langle \lambda, \lambda \rangle \phi^2 + \langle v, v \rangle.$$

Proof. We have

$$\frac{\partial \phi}{\partial t_k} = -\lambda_k \phi, \quad \frac{\partial^2 \phi}{\partial t_k^2} = \lambda_k^2 \phi, \quad \frac{\partial \phi}{\partial x_i} = e^{-\langle \lambda, t \rangle} v_i, \quad \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0.$$

Thus,

$$\tau(\phi) = \sum_k (\lambda_k^2 + \omega_k \lambda_k) \phi = \langle \lambda, \lambda + \omega \rangle \phi,$$

as well as

$$\begin{aligned} \kappa(\phi, \phi) &= \sum_k \lambda_k^2 \phi^2 + e^{-2\langle \lambda, t \rangle} \sum_{ij} (\mu(t) \mu(t)^\mathsf{T})_{ij} v_i v_j \\ &= \langle \lambda, \lambda \rangle \phi^2 + e^{-2\langle \lambda, t \rangle} \langle \mu(t)^\mathsf{T} v, \mu(t)^\mathsf{T} v \rangle \\ &= \langle \lambda, \lambda \rangle \phi^2 + \langle v, v \rangle, \end{aligned}$$

where the final equality follows since v is an eigenvector of $\mu(t)^\mathsf{T} = \text{Exp}(\sum_k A_k^\mathsf{T} t_k)$ with eigenvalue $e^{\langle \lambda, t \rangle}$ by assumption. $\square$

In order to apply Theorem 3.2.2 more effectively, we consider several different cases depending on the properties of the eigenvector v and the corresponding eigenvalues. It turns out that it is most convenient to separately consider the cases when the eigenvector v is isotropic or non-isotropic, since this affects the formula for the conformality operator κ as seen in Lemma 5.4.10.

Let us first treat the case when the eigenvector v is non-isotropic i.e. $\langle v, v \rangle \neq 0$. Here we will need to take square roots of some complex numbers to simplify the formulae. As these complex numbers are constants, the fact that the square root function is not (continuously) defined on the entire complex plane does not play a major role, so any choice of square root will do the trick. To fix a definite choice, we define the the square of any complex number z by

$$\sqrt{z} = \begin{cases} \exp(\frac{1}{2} \log(z)), & z \in \mathbb{C} \setminus (-\infty, 0] \\ i\sqrt{|z|}, & z \in (-\infty, 0] \end{cases}.$$

Note that if the eigenvector v is non-isotropic, this allows us to assume without loss of generality that it satisfies $\langle v, v \rangle = 1$, for if not, we can replace it by $\frac{1}{\sqrt{\langle v, v \rangle}} v$.

Proposition 5.4.11. *Let v be a common eigenvector of the commuting family $\mathcal{A}^T = (A_k^T)_k$, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the vector consisting of the corresponding eigenvalues. Suppose further that the eigenvector v is non-isotropic with $\langle v, v \rangle = 1$.*

(i) *If λ is isotropic, define $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by*

$$\phi(t, x) = e^{-\langle \lambda, t \rangle} \langle v, x \rangle,$$

and define the holomorphic functions $f_p : \mathbb{C} \rightarrow \mathbb{C}$ by

$$\begin{aligned} f_1(z) &= c_1 \int^z e^{-\frac{1}{2} \langle \lambda, \omega \rangle \zeta^2} d\zeta + c_2, \\ f_p(z) &= \int^z e^{-\frac{1}{2} \langle \lambda, \omega \rangle \eta^2} \int^\eta e^{\frac{1}{2} \langle \lambda, \omega \rangle \zeta^2} f_{p-1}(\zeta) d\zeta d\eta, \end{aligned}$$

for arbitrary $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$. Then the composition $f_p \circ \phi$ is proper p -harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ for all $p \geq 1$.

(ii) *If λ is non-isotropic, put*

$$\hat{\mathbb{C}} = \mathbb{C} \setminus ((-i\infty, -i] \cup [i, i\infty)),$$

define $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by

$$\phi(t, x) = \sqrt{\langle \lambda, \lambda \rangle} e^{-\langle \lambda, t \rangle} \langle v, x \rangle,$$

and define the holomorphic functions $f_p : \hat{\mathbb{C}} \rightarrow \mathbb{C}$ by

$$\begin{aligned} f_1(z) &= c_1 \int^z (\zeta^2 + 1)^{-\frac{1}{2} - \frac{\langle \lambda, \omega \rangle}{2\langle \lambda, \lambda \rangle}} d\zeta + c_2, \\ f_p(z) &= \int^z (\eta^2 + 1)^{-\frac{1}{2} - \frac{\langle \lambda, \omega \rangle}{2\langle \lambda, \lambda \rangle}} \int^\eta (\zeta^2 + 1)^{-\frac{1}{2} + \frac{\langle \lambda, \omega \rangle}{2\langle \lambda, \lambda \rangle}} f_{p-1}(\zeta) d\zeta d\eta, \end{aligned}$$

for arbitrary $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$. Then the composition $f_p \circ \phi$ is proper p -harmonic on its open domain $\phi^{-1}(\hat{\mathbb{C}})$ in $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ for all $p \geq 1$.

Proof. Since the tension field τ is linear and the conformality operator κ is bilinear, we see by Lemma 5.4.10 that in the respective cases

$$\begin{aligned} \text{(i)} \quad \tau(\phi) &= \langle \lambda, \omega \rangle \phi, & \kappa(\phi, \phi) &= 1, \\ \text{(ii)} \quad \tau(\phi) &= \langle \lambda, \lambda + \omega \rangle \phi, & \kappa(\phi, \phi) &= \langle \lambda, \lambda \rangle (\phi^2 + 1), \end{aligned}$$

and the claimed result then follows from Theorem 3.2.2. $\square$

Remark 5.4.12. The function f_1 can in case (i) be expressed in terms of the error function, and in case (ii) in terms of the hypergeometric function. The functions f_p for $p > 1$ do not seem to have a closed form in terms of elementary functions in general. However, under the additional assumption that $\langle \lambda, \omega \rangle = 0$, one can easily calculate that in case (i),

$$f_p(z) = c_1 z^{2p-1} + c_2 z^{2p-2},$$

while in case (ii),

$$f_p(z) = c_1 \operatorname{arsinh}(z)^{2p-1} + c_2 \operatorname{arsinh}(z)^{2p-2},$$

where the constants c_1, c_2 are appropriately modified. Note that if $\lambda = 0$, then $\langle \lambda, \lambda \rangle = \langle \lambda, \omega \rangle = 0$, so that case (i) of the proposition implies that the function

$$\psi(x) = c_1 \langle v, x \rangle^{2p-1} + c_2 \langle v, x \rangle^{2p-2}$$

is a proper p -harmonic function on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ which is independent of t , and satisfies the condition of Proposition 5.4.2.

Example 5.4.13 ($G_{4.1}$). The family $\mathcal{A}^T$ corresponding to the Lie group $G_{4.1} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$ consists of the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

All of its eigenvalues are 0, and the eigenspace is one-dimensional and spanned by the unit vector e_3 . Thus case (i) of Proposition 5.4.11 shows that the function

$$x \mapsto a_1 x_3^{2p-1} + a_2 x_3^{2p-2}$$

is proper p -harmonic on $G_{4.1}$ (see also Remark 5.4.12). By construction, this function also satisfies the condition from Proposition 5.4.2. Hence, if p, r, q are positive integers such that $r + q - 1 = p$, Proposition 5.4.2 combined with Proposition 5.4.7 implies that the function defined by

$$(t, x) \mapsto (c_1 t^{2r-1} + c_2 t^{2r-2})(a_1 x_3^{2q-1} + a_2 x_3^{2q-2})$$

is proper p -harmonic on $G_{4.1}$. $\triangle$

Example 5.4.14 ($G_{3.4}^{-1} = \mathbf{Sol}^3$). The family $\mathcal{A}^T$ corresponding to the Lie group $G_{3.4}^\alpha = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^2$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

Its eigenvalues are 1 and α , and the corresponding eigenvectors are the unit vectors e_1 and e_2 , respectively. Consider now the particular case $\alpha = -1$. In this case the Riemannian Lie group $G_{3.4}^{-1}$ becomes the famous Thurston geometry $\mathbf{Sol}^3$. Examples of proper p -harmonic functions on this Lie group have already been constructed in [19, 26]. We can obtain new examples of proper p -harmonic functions on this Lie group by applying case (ii) of Proposition 5.4.11. Indeed, we see that the functions defined by

$$\begin{aligned} (t, x) &\mapsto c_1 \operatorname{arsinh}(e^{-t} x_1)^{2p-1} + c_2 \operatorname{arsinh}(e^{-t} x_1)^{2p-2}, \\ (t, x) &\mapsto c_1 \operatorname{arsinh}(e^t x_2)^{2p-1} + c_2 \operatorname{arsinh}(e^t x_2)^{2p-2}, \end{aligned}$$

are proper p -harmonic on $\mathbf{Sol}^3$ for any $c \in \mathbb{C}^2 \setminus \{0\}$ (see also Remark 5.4.12). $\triangle$

Now let us treat the case when the eigenvector v is isotropic. This case turns out to be richer, as we will now see. To begin with, we see immediately from Lemma 5.4.10 that in this case ϕ will be an eigenfunction in the sense of Definition 4.1.1. In fact, in this case it is no longer essential that the vector λ consists of the eigenvalues, but can rather be chosen arbitrarily.

Proposition 5.4.15. *Let v be a common eigenvector of the commuting family $\mathcal{A}^T = (A_k^T)_k$ and suppose that v is isotropic. Then for any complex vector $\nu \in \mathbb{C}^m$, the function $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ defined by*

$$\phi(t, x) = e^{-\langle \nu, t \rangle} \langle v, x \rangle$$

is an eigenfunction on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ satisfying

$$\tau(\phi) = \langle \nu, \nu + \omega \rangle \phi \quad \text{and} \quad \kappa(\phi, \phi) = \langle \nu, \nu \rangle \phi^2.$$

Proof. The calculation is the same as in the proof of Lemma 5.4.10. $\square$

Thus, one may combine Proposition 5.4.15 with Theorem 4.1.3 to obtain interesting examples of proper p -harmonic functions.

Example 5.4.16 ($G_{3,3}$). The family $\mathcal{A}^T$ corresponding to the Lie group $G_{3,3} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^2$ is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Both of its eigenvalues are 1, and the corresponding eigenvectors are the canonical unit vectors e_1, e_2 . Thus, for any $\zeta, \omega \in \mathbb{C}$, the vector $v = (\zeta, \omega) = \zeta e_1 + \omega e_2$ is also an eigenvector with eigenvalue 1. Furthermore if we require that $\zeta^2 + \omega^2 = 0$ (e.g. $\zeta = 1, \omega = i$), we see that v becomes isotropic, so that Proposition 5.4.15 implies that for any $\nu \in \mathbb{C}$, the function

$$\phi(t, x) = e^{-\nu t} (\zeta x_1 + \omega x_2)$$

is an eigenfunction on $G_{3,3}$ satisfying

$$\tau(\phi) = \nu(\nu + 2) \phi \quad \text{and} \quad \kappa(\phi, \phi) = \nu^2 \phi^2. \quad \triangle$$

Example 5.4.17 ($G_{4,6}^{\alpha\beta}$). The family $\mathcal{A}^T$ corresponding to the Lie group $G_{4,6}^{\alpha\beta} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$ consists of the matrix

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & -1 \\ 0 & 1 & \beta \end{bmatrix}.$$

Its complex eigenvectors are $v^{\pm} = (0, 1, \pm i)$. It is clear that $v^{\pm}$ are isotropic, and so by Proposition 5.4.15, the corresponding functions

$$\phi^{\pm}(t, x) = e^{-\nu t} (x_2 \pm i x_3)$$

are eigenfunctions on $G_{4,6}^{\alpha\beta}$ with

$$\tau(\phi^{\pm}) = \nu(\nu + \alpha + 2\beta) \phi^{\pm} \quad \text{and} \quad \kappa(\phi^{\pm}, \phi^{\pm}) = \nu^2 (\phi^{\pm})^2,$$

for any choice of $\nu \in \mathbb{C}$. $\triangle$

Example 5.4.18 ($G_{4,10}$). The family $\mathcal{A}^T$ corresponding to the Lie group $G_{4,10} = \mathbb{R}^2 \ltimes_{\mathcal{A}} \mathbb{R}^2$ consists of the matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Their common eigenvectors are $v^\pm = (1, \pm i)$. Since v is isotropic, we see from Proposition 5.4.15 that for any $\nu \in \mathbb{C}^2$, the functions

$$\phi^\pm(t, x) = e^{-(\nu_1 t_1 + \nu_2 t_2)}(x_1 \pm ix_2)$$

are eigenfunctions on $G_{4,10}$ with

$$\tau(\phi^\pm) = (\nu_1^2 - 2\nu_1 + \nu_2^2) \phi^\pm \quad \text{and} \quad \kappa(\phi^\pm, \phi^\pm) = (\nu_1^2 + \nu_2^2) (\phi^\pm)^2. \quad \triangle$$

It is quite interesting that in the case when the eigenvector v is isotropic, we can also obtain a construction analogous to Proposition 5.4.11. For this we first observe the following.

Lemma 5.4.19. *Let v be a common eigenvector of the commuting family $\mathcal{A}^T = (A_k^T)_k$, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the vector consisting of the corresponding eigenvalues. Suppose further that the eigenvector v is isotropic and define the function $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by*

$$\phi(t, x) = e^{-\langle \Re \lambda, t \rangle} \langle v, x \rangle.$$

Then the real part $\Re \phi$ and the imaginary part $\Im \phi$ of ϕ are isoparametric functions on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ with

$$\begin{aligned} \tau(\Re \phi) &= \langle \Re \lambda, \Re \lambda + \omega \rangle \Re \phi, \\ \kappa(\Re \phi, \Re \phi) &= \langle \Re \lambda, \Re \lambda \rangle (\Re \phi)^2 + \frac{1}{2} \langle v, \bar{v} \rangle, \end{aligned}$$

and

$$\begin{aligned} \tau(\Im \phi) &= \langle \Re \lambda, \Re \lambda + \omega \rangle \Im \phi, \\ \kappa(\Im \phi, \Im \phi) &= \langle \Re \lambda, \Re \lambda \rangle (\Im \phi)^2 + \frac{1}{2} \langle v, \bar{v} \rangle. \end{aligned}$$

Proof. We perform the calculations only for the real part $\Re \phi$ of ϕ . The calculations for the imaginary part are entirely similar. We have

$$\begin{aligned} \frac{\partial(\Re \phi)}{\partial t_k} &= -(\Re \lambda)_k (\Re \phi), & \frac{\partial^2(\Re \phi)}{\partial t_k^2} &= (\Re \lambda)_k^2 (\Re \phi), \\ \frac{\partial(\Re \phi)}{\partial x_i} &= e^{-\langle \Re \lambda, t \rangle} (\Re v)_i, & \frac{\partial^2(\Re \phi)}{\partial x_i \partial x_j} &= 0. \end{aligned}$$

The formula for $\tau(\Re \phi)$ thus follows immediately. Furthermore, we see that

$$\begin{aligned} \kappa(\Re \phi, \Re \phi) &= \sum_{k=1}^m (\Re \lambda)_k^2 (\Re \phi)^2 + e^{-2\langle \Re \lambda, t \rangle} \sum_{i,j=1}^n (\mu(t) \mu(t)^T)_{ij} (\Re v)_i (\Re v)_j \\ &= \langle \Re \lambda, \Re \lambda \rangle (\Re \phi)^2 + e^{-2\langle \Re \lambda, t \rangle} \langle \mu(t)^T \Re v, \mu(t)^T \Re v \rangle, \end{aligned}$$

as well as

$$\begin{aligned}
& e^{-2\langle \Re \lambda, t \rangle} \langle \mu(t)^T \Re v, \mu(t)^T \Re v \rangle \\
&= \frac{1}{4} e^{-2\langle \Re \lambda, t \rangle} \langle \mu(t)^T (v + \bar{v}), \mu(t)^T (v + \bar{v}) \rangle \\
&= \frac{1}{2} e^{-2\langle \Re \lambda, t \rangle} \left(\Re \left(e^{2\langle \lambda, t \rangle} \langle v, v \rangle \right) + e^{\langle \lambda + \bar{\lambda}, t \rangle} \langle v, \bar{v} \rangle \right) = \frac{1}{2} \langle v, \bar{v} \rangle,
\end{aligned}$$

where the final equality follows since v is assumed to be isotropic. This proves the claim. $\square$

Now we can use the real and imaginary parts of ϕ in conjunction with Theorem 3.2.2 to obtain proper p -harmonic functions. Note that the real and imaginary parts of ϕ are automatically real-valued, so that the functions f_p from Theorem 3.2.2 can be assumed to be functions of a real variable. Similarly as before, we can in this case arrange that the eigenvector v satisfies $\langle v, \bar{v} \rangle = 2$.

Proposition 5.4.20. *Let v be a common eigenvector of the commuting family $\mathcal{A}^T = (A_k^T)_k$, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the vector consisting of the corresponding eigenvalues. Suppose further that the eigenvector v is isotropic and $\langle v, \bar{v} \rangle = 2$.*

(i) *If $\Re \lambda = 0$, define $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by*

$$\phi(x) = \langle v, x \rangle,$$

and define the functions $f_p : \mathbb{R} \rightarrow \mathbb{C}$ by

$$f_p(s) = c_1 s^{2p-1} + c_2 s^{2p-2},$$

for arbitrary $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$.

(ii) *If $\Re \lambda \neq 0$, define $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by*

$$\phi(t, x) = \sqrt{\langle \Re \lambda, \Re \lambda \rangle} e^{-\langle \Re \lambda, t \rangle} \langle v, x \rangle,$$

and define the functions $f_p : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\begin{aligned}
f_1(s) &= c_1 \int^s (\zeta^2 + 1)^{-\frac{1}{2} - \frac{\langle \Re \lambda, \omega \rangle}{2\langle \Re \lambda, \Re \lambda \rangle}} d\zeta + c_2, \\
f_p(s) &= \int^s (\eta^2 + 1)^{-\frac{1}{2} - \frac{\langle \Re \lambda, \omega \rangle}{2\langle \Re \lambda, \Re \lambda \rangle}} \int^\eta (\zeta^2 + 1)^{-\frac{1}{2} + \frac{\langle \Re \lambda, \omega \rangle}{2\langle \Re \lambda, \Re \lambda \rangle}} f_{p-1}(\zeta) d\zeta d\eta,
\end{aligned}$$

for arbitrary $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$.

Then in both cases, the compositions $f_p \circ (\Re \phi)$ and $f_p \circ (\Im \phi)$ are proper p -harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ for all $p \geq 1$.

Proof. Since the tension field τ is linear and the conformality operator κ is bilinear, we see by Lemma 5.4.19 that in the respective cases

- (i) $\tau(\Re \phi) = 0, \quad \kappa(\Re \phi, \Re \phi) = 1,$
- (ii) $\tau(\Re \phi) = \langle \Re \lambda, \Re \lambda + \omega \rangle \Re \phi, \quad \kappa(\Re \phi, \Re \phi) = \langle \Re \lambda, \Re \lambda \rangle ((\Re \phi)^2 + 1),$

and analogous identities hold for $\Im \phi$. The result then follows from Theorem 3.2.2. $\square$

Example 5.4.21 ($G_{4,9}^\alpha$). The family $\mathcal{A}^T$ corresponding to the Lie group $G_{4,9}^\alpha = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3$ consists of the matrix

$$\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}.$$

Its eigenvectors are $v^\pm = (1, \pm i)$ and the corresponding eigenvalues are $\lambda^\pm = \alpha \pm i$. Note that the eigenvectors $v^\pm$ are isotropic, so that Proposition 5.4.11 may be applied. We consider only the eigenvector $v = (1, i)$ and its eigenvalue $\lambda = \alpha + i$ for simplicity.

In the case $\alpha = 0$, we have $\Re \lambda = 0$, and so it follows from case (i) of Proposition 5.4.11 that, for any positive integer p and any $c \in \mathbb{C}^2 \setminus \{0\}$, the functions

$$\begin{aligned} (t, \xi, x) &\mapsto c_1 x_1^{2p-1} + c_2 x_1^{2p-2}, \\ (t, \xi, x) &\mapsto c_1 x_2^{2p-1} + c_2 x_2^{2p-2}, \end{aligned}$$

are proper p -harmonic on $G_{4,9}^0$. By construction, these functions also satisfy the condition from Proposition 5.4.2, so we may also multiply them by a function of t to obtain even more examples of proper p harmonic functions, cf. Example 5.4.13.

In the case $\alpha \neq 0$, we have $\Re \lambda \neq 0$, so that we can apply case (ii) of Proposition 5.4.11. However, finding a general formula for f_p seems unfeasible. Let us at least note that Proposition 5.4.20 implies that, for any $c \in \mathbb{C}^2 \setminus \{0\}$, the functions given by

$$\begin{aligned} (t, \xi, x) &\mapsto c_1 \frac{2(\alpha e^{-\alpha t} x_1)^3 + 3\alpha e^{-\alpha t} x_1}{\{(\alpha e^{-\alpha t} x_1)^2 + 1\}^{3/2}} + c_2, \\ (t, \xi, x) &\mapsto c_1 \frac{2(\alpha e^{-\alpha t} x_2)^3 + 3\alpha e^{-\alpha t} x_2}{\{(\alpha e^{-\alpha t} x_2)^2 + 1\}^{3/2}} + c_2, \end{aligned}$$

are proper harmonic on $G_{4,9}^\alpha$, and the functions given by

$$\begin{aligned} (t, \xi, x) &\mapsto c_1 \left(\operatorname{arsinh}(\alpha e^{-\alpha t} x_1) + \frac{(\alpha e^{-\alpha t} x_1)^3}{3 \{(\alpha e^{-\alpha t} x_1)^2 + 1\}^{3/2}} \right) \\ &\quad + c_2 \left(\frac{2(\alpha e^{-\alpha t} x_1)^3 + 3\alpha e^{-\alpha t} x_1}{\{(\alpha e^{-\alpha t} x_1)^2 + 1\}^{3/2}} \operatorname{arsinh}(\alpha e^{-\alpha t} x_1) - \frac{1}{(\alpha e^{-\alpha t} x_1)^2 + 1} \right), \\ (t, \xi, x) &\mapsto c_1 \left(\operatorname{arsinh}(\alpha e^{-\alpha t} x_2) + \frac{(\alpha e^{-\alpha t} x_2)^3}{3 \{(\alpha e^{-\alpha t} x_2)^2 + 1\}^{3/2}} \right) \\ &\quad + c_2 \left(\frac{2(\alpha e^{-\alpha t} x_2)^3 + 3\alpha e^{-\alpha t} x_2}{\{(\alpha e^{-\alpha t} x_2)^2 + 1\}^{3/2}} \operatorname{arsinh}(\alpha e^{-\alpha t} x_2) - \frac{1}{(\alpha e^{-\alpha t} x_2)^2 + 1} \right), \end{aligned}$$

are proper biharmonic on $G_{4,9}^\alpha$. $\triangle$

Appendix A

Technical Details

A.1 Some matrix identities

In this appendix we will prove the matrix identities used in Section 4.2. As in Section 4.2, we let E_{ij} denote the $n \times n$ matrix satisfying

$$(E_{ij})_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta},$$

and we also define the $n \times n$ matrices

$$X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}), \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}), \quad D_t = E_{tt},$$

We begin by observing that

$$\begin{aligned} (E_{ij} E_{kl})_{\alpha\beta} &= \sum_{t=1}^n (E_{ij})_{\alpha t} (E_{kl})_{t\beta} \\ &= \sum_{t=1}^n \delta_{i\alpha} \delta_{jt} \delta_{kt} \delta_{l\beta} \\ &= \delta_{i\alpha} \delta_{jk} \delta_{l\beta} = \delta_{jk} (E_{il})_{\alpha\beta}, \end{aligned}$$

so that

$$E_{ij} E_{kl} = \delta_{jk} E_{il}. \tag{A.1}$$

It follows also that

$$\begin{aligned} (E_{ij} E_{kl} E_{ab})_{\alpha\beta} &= \sum_{t=1}^n (E_{ij} E_{kl})_{\alpha t} (E_{ab})_{t\beta} \\ &= \sum_{t=1}^n \delta_{i\alpha} \delta_{jk} \delta_{lt} \delta_{at} \delta_{b\beta} \\ &= \delta_{i\alpha} \delta_{jk} \delta_{la} \delta_{b\beta} = \delta_{jk} \delta_{la} (E_{ib})_{\alpha\beta}, \end{aligned}$$

so that

$$E_{ij} E_{kl} E_{ab} = \delta_{jk} \delta_{la} E_{ib}. \tag{A.2}$$

The identity (A.1) implies that

$$\begin{aligned} X_{rs}^2 &= \frac{1}{2}(E_{rs}^2 + E_{rs} E_{sr} + E_{sr} E_{rs} + E_{sr}^2) \\ &= \frac{1}{2}(\delta_{rs} E_{rs} + E_{rr} + E_{ss} + \delta_{sr} E_{sr}) \\ &= \frac{1}{\sqrt{2}} \delta_{rs} X_{rs} + \frac{1}{2}(D_r + D_s), \end{aligned}$$

and analogously

$$Y_{rs}^2 = \frac{1}{\sqrt{2}} \delta_{rs} X_{rs} - \frac{1}{2}(D_r + D_s).$$

This implies that

$$\sum_{1 \leq r < s \leq n} X_{rs}^2 = \frac{1}{2} \sum_{1 \leq r < s \leq n} (D_r + D_s) = \frac{n-1}{2} I_n,$$

as well as

$$\sum_{1 \leq r < s \leq n} Y_{rs}^2 = -\frac{1}{2} \sum_{1 \leq r < s \leq n} (D_r + D_s) = \frac{1-n}{2} I_n,$$

Note that this gives in particular that

$$\begin{aligned} \sum_{1 \leq r < s \leq n} Y_{rs}^2 - \sum_{1 \leq r < s \leq n} X_{rs}^2 - \sum_{t=1}^n D_t^2 \\ = \frac{1-n}{2} I_n - \frac{n-1}{2} I_n - I_n = -n I_n. \end{aligned}$$

We continue by observing that (A.2) implies

$$\begin{aligned} X_{rs} E_{jl} X_{rs}^T &= \frac{1}{2}(E_{rs} + E_{sr}) E_{jl} (E_{sr} + E_{rs}) \\ &= \frac{1}{2}(E_{rs} E_{jl} E_{sr} + E_{rs} E_{jl} E_{rs} + E_{sr} E_{jl} E_{sr} + E_{sr} E_{jl} E_{rs}) \\ &= \frac{1}{2}(\delta_{sj} \delta_{ls} E_{rr} + \delta_{sj} \delta_{lr} E_{rs} + \delta_{rj} \delta_{ls} E_{sr} + \delta_{rj} \delta_{lr} E_{ss}) \\ &= \frac{1}{2}(\delta_{jl} \delta_{sj} D_r + \delta_{sj} \delta_{lr} E_{lj} + \delta_{rj} \delta_{ls} E_{lj} + \delta_{jl} \delta_{rj} D_s) \\ &= \frac{1}{2} \delta_{jl} (\delta_{sj} D_r + \delta_{rj} D_s) + \frac{1}{2} (\delta_{sj} \delta_{lr} + \delta_{rj} \delta_{ls}) E_{lj}, \end{aligned}$$

and in an entirely similar way

$$Y_{rs} E_{jl} Y_{rs}^T = \frac{1}{2} \delta_{jl} (\delta_{sj} D_r + \delta_{rj} D_s) - \frac{1}{2} (\delta_{sj} \delta_{lr} + \delta_{rj} \delta_{ls}) E_{lj}.$$

Hence,

$$\begin{aligned} \sum_{1 \leq r < s \leq n} X_{rs} E_{jl} X_{rs}^T &= \frac{1}{2} \delta_{jl} \sum_{1 \leq r < s \leq n} (\delta_{sj} D_r + \delta_{rj} D_s) + \frac{1}{2} E_{lj} \sum_{1 \leq r < s \leq n} (\delta_{sj} \delta_{lr} + \delta_{rj} \delta_{ls}) \\ &= \frac{1}{2} \delta_{jl} (I_n - D_j) + \frac{1}{2} E_{lj} (1 - \delta_{jl}) = \frac{1}{2} \delta_{jl} I_n + \frac{(-1)^{\delta_{jl}}}{2} E_{lj}, \end{aligned}$$

and similarly

$$\begin{aligned}\sum_{1 \leq r < s \leq n} Y_{rs} E_{jl} Y_{rs}^T &= \frac{1}{2} \delta_{jl} \sum_{1 \leq r < s \leq n} (\delta_{sj} D_r + \delta_{rj} D_s) - \frac{1}{2} E_{lj} \sum_{1 \leq r < s \leq n} (\delta_{sj} \delta_{lr} + \delta_{rj} \delta_{ls}) \\ &= \frac{1}{2} \delta_{jl} (I_n - D_j) - \frac{1}{2} E_{lj} (1 - \delta_{jl}) = \frac{1}{2} \delta_{jl} I_n - \frac{1}{2} E_{lj}.\end{aligned}$$

Finally,

$$\sum_{t=1}^n D_t E_{jl} D_t^T = \sum_{t=1}^n E_{tt} E_{jl} E_{tt} = \sum_{t=1}^n \delta_{tj} \delta_{lt} E_{tt} = \delta_{jl} \sum_{t=1}^n \delta_{tj} D_t = \delta_{jl} D_j,$$

which gives

$$\begin{aligned}\sum_{1 \leq r < s \leq n} Y_{rs} E_{jl} Y_{rs}^T - \sum_{1 \leq r < s \leq n} X_{rs} E_{jl} X_{rs}^T - \sum_{t=1}^n D_t E_{jl} D_t^T \\ = \frac{1}{2} \delta_{jl} I_n - \frac{1}{2} E_{lj} - \frac{1}{2} \delta_{jl} I_n - \frac{(-1)^{\delta_{jl}}}{2} E_{lj} - \delta_{jl} D_j = -E_{lj}.\end{aligned}$$

A.2 The Heisenberg algebra $\mathfrak{h}^{2n+1}$

The Heisenberg algebra $\mathfrak{h}^{2n+1}$ is a $(2n+1)$ -dimensional Lie algebra with basis $\{\Xi, X_1, \dots, X_{2n}\}$ whose commutators satisfy

$$[X_{2i-1}, X_{2i}] = \Xi$$

for all $1 \leq i \leq n$, and all the other commutators are 0. An easy calculation [14, Proposition 9] shows that the space $\text{Der}(\mathfrak{h}^{2n+1})$ of derivations of this algebra consists of the endomorphisms (represented in the basis $\{\Xi, X_1, \dots, X_{2n}\}$) of the form

$$\begin{bmatrix} a & v^T \\ 0_{2n \times 1} & A \end{bmatrix} \in \mathbb{R}^{(2n+1) \times (2n+1)},$$

where $a \in \mathbb{R}$ and $v \in \mathbb{R}^{2n}$ are arbitrary, while the matrix $A \in \mathbb{R}^{2n \times 2n}$ is of the block form

$$A = \begin{bmatrix} A_{(1,1)} & -\text{adj}(A_{(2,1)}) & -\text{adj}(A_{(3,1)}) & \dots & -\text{adj}(A_{(n,1)}) \\ A_{(2,1)} & A_{(2,2)} & -\text{adj}(A_{(3,2)}) & \dots & -\text{adj}(A_{(n,2)}) \\ A_{(3,1)} & A_{(3,2)} & A_{(3,3)} & \dots & -\text{adj}(A_{(n,3)}) \\ \vdots & & & \ddots & \vdots \\ A_{(n,1)} & A_{(n,2)} & A_{(n,3)} & \dots & A_{(n,n)} \end{bmatrix},$$

where $A_{(i,j)} \in \mathbb{R}^{2n \times 2n}$ are such that $\text{Tr } A_{(i,i)} = a$ for $1 \leq i \leq n$, and $\text{adj}(A_{(i,j)})$ denotes the adjugate i.e. the transpose of the cofactor matrix of $A_{(i,j)}$. Since these are 2×2 matrices, this means that if we denote

$$A_{(i,j)} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \delta_{ij} \end{bmatrix}, \quad \text{then} \quad \text{adj}(A_{(i,j)}) = \begin{bmatrix} \delta_{ij} & -\beta_{ij} \\ -\gamma_{ij} & \alpha_{ij} \end{bmatrix}.$$

The matrix A possesses the following wonderful property.

Lemma A.2.1. Let $a \in \mathbb{R}$, let $A \in \mathbb{R}^{2n \times 2n}$ be of the form (5.8), and let

$$J_{2n} = \text{diag} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{n \text{ times}} \right) \in \mathbb{R}^{2n \times 2n}.$$

Then the following identity holds

$$J_{2n}^T A = (aI_{2n} - A^T) J_{2n}^T.$$

Proof. We shall prove that

$$J_{2n}^T A + A^T J_{2n}^T = aJ_{2n}^T.$$

Let us denote

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then we see that

$$\begin{aligned} J_{2n}^T A + A^T J_{2n}^T &= \begin{bmatrix} J^T A_{(1,1)} & -J^T \text{adj}(A_{(2,1)}) & \dots & -J^T \text{adj}(A_{(n,1)}) \\ J^T A_{(2,1)} & J^T A_{(2,2)} & \dots & -J^T \text{adj}(A_{(n,2)}) \\ \vdots & & \ddots & \vdots \\ J^T A_{(n,1)} & J^T A_{(n,2)} & \dots & J^T A_{(n,n)} \end{bmatrix} + \\ &+ \begin{bmatrix} A_{(1,1)}^T J^T & A_{(2,1)}^T J^T & \dots & A_{(n,1)}^T J^T \\ -\text{adj}(A_{(2,1)})^T J^T & A_{(2,2)}^T J^T & \dots & A_{(n,2)}^T J^T \\ \vdots & & \ddots & \vdots \\ -\text{adj}(A_{(n,1)})^T J^T & -\text{adj}(A_{(n,2)})^T J^T & \dots & A_{(n,n)}^T J^T \end{bmatrix}. \end{aligned}$$

We thus want to calculate

- (i) $J^T A_{(i,i)} + A_{(i,i)}^T J^T$, $1 \leq i \leq n$, (the diagonal),
- (ii) $-J^T \text{adj}(A_{(i,j)}) + A_{(i,j)}^T J^T$, $1 \leq i < j \leq n$, (above the diagonal),
- (iii) $J^T A_{(i,j)} - \text{adj}(A_{(i,j)})^T J^T$, $1 \leq j < i \leq n$, (below the diagonal).

In all three cases, we will denote

$$A_{(i,j)} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \delta_{ij} \end{bmatrix},$$

where if $i = j$, we additionally have $\delta_{ii} = a - \alpha_{ii}$ since $\text{Tr } A_{(i,i)} = a$. For case (i), we have

$$\begin{aligned} J^T A_{(i,i)} + A_{(i,i)}^T J^T &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{ii} & \beta_{ii} \\ \gamma_{ii} & a - \alpha_{ii} \end{bmatrix} + \begin{bmatrix} \alpha_{ii} & \gamma_{ii} \\ \beta_{ii} & a - \alpha_{ii} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_{ii} & a - \alpha_{ii} \\ -\alpha_{ii} & -\beta_{ii} \end{bmatrix} + \begin{bmatrix} -\gamma_{ii} & \alpha_{ii} \\ -a + \alpha_{ii} & \beta_{ii} \end{bmatrix} = aJ^T. \end{aligned}$$

For case (ii), we have

$$\begin{aligned} -J^T \operatorname{adj}(A_{(i,j)}) + A_{(i,j)}^T J^T &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{ij} & -\beta_{ij} \\ -\gamma_{ij} & \alpha_{ij} \end{bmatrix} + \begin{bmatrix} \alpha_{ij} & \gamma_{ij} \\ \beta_{ij} & \delta_{ij} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= -\begin{bmatrix} -\gamma_{ij} & \alpha_{ij} \\ -\delta_{ij} & \beta_{ij} \end{bmatrix} + \begin{bmatrix} -\gamma_{ij} & \alpha_{ij} \\ -\delta_{ij} & \beta_{ij} \end{bmatrix} = 0. \end{aligned}$$

Finally, for case (iii), we have

$$\begin{aligned} J^T A_{(i,j)} - \operatorname{adj}(A_{(i,j)})^T J^T &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \delta_{ij} \end{bmatrix} - \begin{bmatrix} \delta_{ij} & -\gamma_{ij} \\ -\beta_{ij} & \alpha_{ij} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_{ij} & \delta_{ij} \\ -\alpha_{ij} & -\beta_{ij} \end{bmatrix} - \begin{bmatrix} \gamma_{ij} & \delta_{ij} \\ -\alpha_{ij} & -\beta_{ij} \end{bmatrix} = 0. \end{aligned}$$

Thus we see that

$$J_{2n}^T A + A^T J_{2n}^T = \begin{bmatrix} aJ^T & 0 & \dots & 0 \\ 0 & aJ^T & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & aJ^T \end{bmatrix} = aJ_{2n}^T. \quad \square$$

Corollary A.2.2. *Let $a \in \mathbb{R}$ and $A \in \mathbb{R}^{2n \times 2n}$ be as in (5.8), and let*

$$J_{2n} = \operatorname{diag} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{n \text{ times}} \right) \in \mathbb{R}^{2n \times 2n}.$$

Then

$$J_{2n}^T \operatorname{Exp}(At) = e^{at} (J_{2n} \operatorname{Exp}(-At))^T.$$

Proof. We see by the previous lemma that

$$\begin{aligned} J_{2n}^T \operatorname{Exp}(At) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} J_{2n}^T A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} (aI_{2n} - A^T)^k J_{2n}^T \\ &= e^{at} \operatorname{Exp}(-A^T t) J_{2n}^T = e^{at} (J_{2n} \operatorname{Exp}(-At))^T, \end{aligned}$$

as claimed. $\square$

The identity (5.10) that was used several times in Section 5.3 now easily follows by induction from the corollary above.

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