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Bachelor's thesis

The Paley-Wiener Theorems for Gevrey Functions and Ultradistributions



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Abstract

In this thesis we study the spaces of Gevrey functions and ultradistributions, focusing primarily on the properties reflected on their Fourier-Laplace transforms. In particular, we study the Paley-Wiener Theorems for compactly supported Gevrey functions and compactly supported Gevrey ultradistributions.

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Contents

1. Introduction	1
2. Preliminaries	3
2.1. Real analysis, integration, and the Fourier transform	6
2.2. Complex analysis	9
2.3. Distribution theory	13
3. Gevrey Functions	16
3.1. The spaces $\mathcal{G}^s$ and $\mathcal{G}_c^s$	16
3.2. The Paley-Wiener Theorem for $\mathcal{G}_c^s$	19
4. Gevrey Ultradistributions	28
4.1. The spaces $\mathcal{D}'_s$ and $\mathcal{E}'_s$	28
4.2. The Paley-Wiener Theorem for $\mathcal{E}'_s$	36
5. Discussion	47
5.1. Summary	47
5.2. Future work	48
Bibliography	51
Appendix A. Convergence of an integral	52
Appendix B. Estimate of a modulus	53

1. Introduction

The Fourier transform of a function $f \in \mathcal{L}^1(\mathbb{R}^d)$ is defined as

$$\widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx, \quad \xi \in \mathbb{R}^d, \quad (1.1)$$

where $\langle x, \xi \rangle = \sum_{j=1}^d x_j \xi_j$. A natural question that arises is which properties of functions can be retrieved from their Fourier transforms. One of the most basic such results is the Inversion Theorem, which states that an $\mathcal{L}^1$ function whose Fourier transform is also $\mathcal{L}^1$ can be recovered a.e. from the Fourier transform.

In 1934, Paley and Wiener [PW34] presented results which give a relation between the support of a function and the holomorphy of its Fourier-Laplace transform, which is obtained by replacing $\xi \in \mathbb{R}^d$ in (1.1) by a complex vector $\zeta \in \mathbb{C}^d$. In their honour, any such result is usually referred to as a Paley-Wiener Theorem.

One of the results that they proved was that an $\mathcal{L}^2(\mathbb{R})$ function is essentially supported on the positive real axis if and only if its Fourier-Laplace transform is well-defined, holomorphic, and square integrable over horizontal lines in the lower complex half-plane.

Another result states that an $\mathcal{L}^2(\mathbb{R})$ function is essentially supported in $[-R, R]$ if and only if its Fourier-Laplace transform Ψ is entire and of exponential type R , in that it satisfies $|\Psi(\zeta)| \leq C e^{R|\zeta|}$, for every $\zeta \in \mathbb{C}$, and some constant $C > 0$.

A generalization was later presented by Schwartz in [Sch57]. In his work he extended the notion of a function to a more general concept of a distribution, namely he worked with the spaces $\mathcal{D}'$ and $\mathcal{E}'$, defined as the sets of all continuous linear functionals on $\mathcal{C}_c^\infty$ and $\mathcal{C}^\infty$ respectively. He also proved theorems of Paley-Wiener type for the spaces $\mathcal{C}_c^\infty$ and $\mathcal{E}'$. These theorems state that [Hör90, Theorem 7.3.1] a function $f \in \mathcal{C}_c^\infty$ is supported in a compact convex set $K \subseteq \mathbb{R}^d$ if and only if its Fourier-Laplace transform Ψ is an entire function that satisfies for each $N > 0$ and some constant $C_N > 0$ the estimate

$$|\Psi(\zeta)| \leq C_N (1 + |\zeta|)^{-N} e^{H_K(\operatorname{Im} \zeta)}, \quad \zeta \in \mathbb{C}^d, \quad (1.2)$$

where H_K is the supporting function of K defined as $H_K(y) = \sup_{x \in K} \langle x, y \rangle$, whereas a distribution $u \in \mathcal{E}'$ is supported in a compact convex set $K \subseteq \mathbb{R}^d$ if and only if its Fourier-Laplace transform Ψ is an entire function that satisfies for some constants $C, N > 0$ the estimate

$$|\Psi(\zeta)| \leq C (1 + |\zeta|)^N e^{H_K(\operatorname{Im} \zeta)}, \quad \zeta \in \mathbb{C}^d. \quad (1.3)$$

In this work we are interested in a more specific class of functions called the Gevrey class $\mathcal{G}^s$ of order $s \geq 1$, defined as the space of all smooth functions f that satisfy for every compact set K and some constant $\tau = \tau_K > 0$ the estimate

$$\sup_{x \in K} |\partial^\alpha f(x)| \lesssim \tau^{|\alpha|} \alpha!^s, \quad \alpha \in \mathbb{N}^d.$$

The Gevrey classes serve as intermediate spaces between the spaces of real-analytic functions and smooth functions, thus providing a way of categorizing the regularity of functions.

We will also work with the spaces $\mathcal{D}'_s$ and $\mathcal{E}'_s$ of Gevrey ultradistributions, defined as the sets of continuous linear functionals on $\mathcal{G}_c^s$ and $\mathcal{G}^s$ respectively. These spaces are in fact strictly larger than their classical counterparts $\mathcal{D}'$ and $\mathcal{E}'$, since the spaces of Gevrey test functions are strict subsets of the spaces of classical test functions.

Our main goal is to present the proofs of the Paley-Wiener Theorems for compactly supported Gevrey functions and ultradistributions. The main difference between the Gevrey versions and the classical versions of the theorems is that the polynomial decay and growth in the estimates (1.2) and (1.3) are replaced by exponential decay and growth respectively. More precisely, a function $f \in \mathcal{G}_c^s$ is supported in a compact convex set $K \subseteq \mathbb{R}^d$ if and only if its Fourier-Laplace transform Ψ is an entire function that satisfies for some constants $C, \lambda > 0$ the estimate

$$|\Psi(\zeta)| \leq C \exp \left(-\lambda |\zeta|^{\frac{1}{s}} + H_K(\operatorname{Im} \zeta) \right), \quad \zeta \in \mathbb{C}^d,$$

whereas an ultradistribution $u \in \mathcal{E}'_s$ is supported in a compact convex set $K \subseteq \mathbb{R}^d$ if and only if its Fourier-Laplace transform Ψ is an entire function that satisfies for every $\lambda > 0$ and some $C_\lambda > 0$ the estimate

$$|\Psi(\zeta)| \leq C_\lambda \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_K(\operatorname{Im} \zeta) \right), \quad \zeta \in \mathbb{C}^d.$$

The strenghtening (resp. weakening) of the estimate of the Fourier-Laplace transform is to be expected due to the aforementioned strict inclusions $\mathcal{G}_c^s \subsetneq \mathcal{C}_c^\infty$ and $\mathcal{E}' \subsetneq \mathcal{E}'_s$.

The thesis is structured in the following way: in Chapter 2, we will begin by recalling some concepts and prerequisites needed for the understanding of this paper; in Chapter 3, we will introduce the Gevrey class of functions, discuss some of its properties, and prove the Paley-Wiener Theorem for this class; in Chapter 4, we will define Gevrey ultradistributions, discuss some preparatory results, and then finally prove the Paley-Wiener Theorem for ultradistributions.

2. Preliminaries

Let us begin by presenting the notation and some estimates that will be used throughout this work. We will mostly use standard mathematical notation, so below we only state the notation specific to this work, as well as some notation that the reader might not be familiar with.

The sets of real and natural numbers will be denoted by $\mathbb{R}$ and $\mathbb{N}$ respectively, and $\mathbb{R}_+$ and $\mathbb{N}_+$ will denote the strictly positive real and natural numbers respectively. If not otherwise specified, we will usually use the letter t to denote a number in $\mathbb{R}_+$ and the letter n to denote a number in $\mathbb{N}_+$. The letter d will always denote the dimension of a given space. In $\mathbb{R}^d$, we denote the open ball with radius $R > 0$ centered at the origin by B_R , i.e.

$$B_R = \{x \in \mathbb{R}^d : |x| < R\}.$$

The closed ball will be denoted by $\overline{B}_R$.

We will mostly use the greek letter ζ to denote a complex vector in $\mathbb{C}^d$. The greek letters ξ and η will represent the real and the imaginary parts of ζ respectively. By this we mean that if

$$\zeta = (\zeta_1, \dots, \zeta_d) = (\xi_1 + i\eta_1, \dots, \xi_d + i\eta_d) \in \mathbb{C}^d, \quad \text{then}$$

$$\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \quad \text{and} \quad \eta = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d.$$

The letter z will usually denote a complex number in $\mathbb{C}$ and will most often be used as a temporary variable. Its real and imaginary parts will usually be denoted by x and y respectively. Given two complex vectors $\nu, \zeta \in \mathbb{C}^d$, we define their scalar product by

$$\langle \nu, \zeta \rangle = \sum_{j=1}^d \overline{\nu}_j \zeta_j.$$

The conjugation is taken on the first variable since the first variable will be almost always be real throughout this paper. The complex upper half plane will be denoted by

$$\mathbb{C}_+ = \{\zeta \in \mathbb{C} : \eta > 0\},$$

and $\overline{\mathbb{C}}_+$ will denote its closure $\mathbb{C}_+ \cup \mathbb{R}$.

If x belongs to the intersection of the domains of two functions f and g , then the notation $f(x) \lesssim g(x)$ will mean that $f(x) \leq Cg(x)$ for some constant $C > 0$. This will

make the estimates more readable and will allow us to focus only on the important factors of the estimates.

When dealing with products of the form $\prod_{j=1}^n x_j$, we define the product to be equal to 1 if $n = 0$ to avoid treating this special case separately every time.

If $F(\Omega)$ is a function space, the domain space Ω will usually be omitted in case that Ω is the whole $\mathbb{R}^d$ or $\mathbb{C}^d$ or when Ω is obvious from the context, meaning that we will simply write F .

The results we present in this work will generally concern multivariable functions, so that the multi-index notation will be used extensively. By multi-index notation, we mean that, if $\zeta \in \mathbb{C}^d$ and $\alpha, \beta \in \mathbb{N}^d$, we define

$$\begin{aligned}\beta &:= \alpha, \quad \text{if } \beta_j \leq \alpha_j \text{ for every } 1 \leq j \leq d, \\ \alpha \pm \beta &:= (\alpha_1 \pm \beta_1, \dots, \alpha_d \pm \beta_d), \\ \alpha! &:= \alpha_1! \dots \alpha_d!, \\ \binom{\alpha}{\beta} &:= \frac{\alpha!}{\beta!(\alpha - \beta)!}, \\ |\alpha| &:= \alpha_1 + \dots + \alpha_d, \\ \zeta^\alpha &:= \zeta_1^{\alpha_1} \dots \zeta_d^{\alpha_d}, \quad \text{and} \\ \partial^\alpha &= \frac{\partial^\alpha}{\partial x^\alpha} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}.\end{aligned}$$

The letters α and β will always be multi-indices, even if we do not specify so explicitly. This notation also simplifies formulae such as the multinomial formula

$$(\zeta_1 + \zeta_2 + \dots + \zeta_d)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \zeta^\alpha, \quad (2.1)$$

the integration by parts formula for compactly supported functions

$$\int (\partial^\alpha f) g \, dx = (-1)^{|\alpha|} \int f (\partial^\alpha g) \, dx,$$

and the Leibniz formula

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f) (\partial^\beta g).$$

Let us also recall some standard estimates concerning the multi-index notation. If we put $\zeta_j = 1$ for $1 \leq j \leq d$ in (2.1), we obtain

$$d^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!}. \quad (2.2)$$

Using (2.2), we also find that

$$|\alpha|! = \alpha! \frac{|\alpha|!}{\alpha!} \leq \alpha! \sum_{|\beta|=|\alpha|} \frac{|\alpha|!}{\beta!} = \alpha! d^{|\alpha|}, \quad (2.3)$$

whereas it is obvious that

$$\alpha! \leq |\alpha|! \leq |\alpha|^{|\alpha|}.$$

If $d = 2$, (2.3) implies in particular that

$$(\alpha_1 + \alpha_2)! \leq 2^{\alpha_1 + \alpha_2} \alpha_1! \alpha_2!. \quad (2.4)$$

For general multi-indices, (2.4) implies that

$$(\alpha + \beta)! = \prod_{j=1}^d (\alpha_j + \beta_j)! \leq \prod_{j=1}^d 2^{\alpha_j + \beta_j} \alpha_j! \beta_j! = 2^{|\alpha| + |\beta|} \alpha! \beta!.$$

We also note that

$$\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} = \sum_{\beta \leq \alpha} \prod_{j=1}^d \binom{\alpha_j}{\beta_j} = \prod_{j=1}^d \sum_{\beta_j \leq \alpha_j} \binom{\alpha_j}{\beta_j} = \prod_{j=1}^d 2^{\alpha_j} = 2^{|\alpha|}.$$

We will often want to compare $|\zeta^\alpha|$ and $|\zeta|^n$ when $|\alpha| = n$, so we observe that

$$|\zeta^\alpha| \leq |\zeta|^n$$

is trivial, whereas

$$|\zeta|^n \leq d^n \max_{|\alpha|=n} |\zeta^\alpha| \quad (2.5)$$

follows since $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for all non-negative x and y , so that

$$|\zeta| = \sqrt{\sum_{k=1}^d |\zeta_k|^2} \leq \sum_{k=1}^d |\zeta_k|,$$

and thus the multinomial formula yields

$$|\zeta|^n \leq \left(\sum_{k=1}^d |\zeta_k| \right)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} |\zeta^\alpha| \leq \max_{|\alpha|=n} |\zeta^\alpha| \sum_{|\alpha|=n} \frac{n!}{\alpha!} = d^n \max_{|\alpha|=n} |\zeta^\alpha|.$$

Finally, let us note that for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$, the Taylor series expansion of e^t gives

$$\frac{t^n}{n!} \leq e^t,$$

which implies in particular that

$$\frac{t^{sn}}{n!^s} \leq e^{st},$$

for every $s > 0$.

We now proceed by giving a short review of the topics required for the understanding of this work.

2.1. Real analysis, integration, and the Fourier transform

We will assume that the reader is familiar with the standard concepts and results in real analysis. However, let us recall the following.

2.1 Definition (Differentiability). Let $\Omega \subseteq \mathbb{R}^d$ be open. A map $f : \Omega \rightarrow \mathbb{C}$ is said to be *differentiable* at $x \in \Omega$ if there exists a vector $A \in \mathbb{C}^d$ such that

$$f(x+h) = f(x) + \sum_{j=1}^d A_j h_j + R(h)$$

whenever $x+h \in \Omega$, and where R is a map satisfying

$$\frac{R(h)}{|h|} \rightarrow 0, \quad \text{whenever } h \rightarrow 0.$$

The A_j in the preceding definition can be shown to be the *partial derivatives* of f at x , i.e.

$$\partial_j f(x) = A_j, \quad 1 \leq j \leq d.$$

As usual, we define $\mathcal{C}(\Omega)$ to be the space of continuous functions on Ω , and $\mathcal{C}^k(\Omega)$, $k \geq 1$, to be the space of functions that are continuous on Ω and whose partial derivatives up to order k exist and are also continuous on Ω . We also define the space of *smooth* functions on Ω as

$$\mathcal{C}^\infty(\Omega) := \bigcap_{k \geq 1} \mathcal{C}^k(\Omega).$$

2.2 Taylor's formula. Let $f \in \mathcal{C}^k$. Then for every $x, h \in \mathbb{R}^d$,

$$f(x+h) = \sum_{|\alpha| < k} \frac{h^\alpha}{\alpha!} \partial^\alpha f(x) + k \int_0^1 (1-\theta)^{k-1} \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} \partial^\alpha f(x+\theta h) d\theta.$$

Proof. See e.g. [Hör90, Equation (1.1.7)].

Q.E.D.

2.3 Definition. The *support* of a function in $\mathcal{C}^\infty(\Omega)$ is defined as the complement of the largest open set in which f vanishes, or equivalently,

$$\text{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The space of smooth functions whose support is compact will be denoted by $\mathcal{C}_c^\infty(\Omega)$.

The compact sets throughout this work will often be assumed to be convex. If the set is not convex, then we will sometimes work with its *convex hull*, defined as the

smallest convex set containing the original set. The supporting function of a compact convex set K is defined as

$$H_K(x) := \sup_{t \in K} \langle t, x \rangle.$$

The convexity assumption is important because it unlocks the following result.

2.4 Hyperplane Separation Theorem. *Let $K_1, K_2 \subseteq \mathbb{R}^d$ be two disjoint convex compact sets. Then there exists a real unit vector $v \in \mathbb{R}^d$ and two constants $c_1, c_2 \in \mathbb{R}$ such that $c_2 < c_1$ and*

$$\begin{aligned} \langle x, v \rangle &\geq c_1, & x \in K_1, \\ \langle x, v \rangle &\leq c_2, & x \in K_2. \end{aligned}$$

Proof. See e.g. [BV04, §2.5].

Q.E.D.

Remark. Note that the inequalities imply that

$$-\inf_{x \in K_1} \langle x, v \rangle \leq -c_1, \quad \sup_{x \in K_2} \langle x, v \rangle \leq c_2,$$

and thus

$$H_{K_2}(v) + H_{K_1}(-v) = \sup_{x \in K_2} \langle x, v \rangle - \inf_{x \in K_1} \langle x, v \rangle \leq c_2 - c_1 < 0,$$

for some unit vector $v \in \mathbb{R}^d$.

All integrals throughout this work will be taken in the Lebesgue sense. We note that not a lot of knowledge of the Lebesgue integral is required for the understanding of this work. However, the following two theorems will be used.

2.5 Dominated Convergence Theorem. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of real-valued measurable functions on a measurable set $X \subseteq \mathbb{R}^d$ converging pointwise to some function f . Suppose that there exists a Lebesgue integrable function g such that $|f_n(x)| \leq g(x)$ for all $n \geq 1$ and $x \in X$. Then*

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| dx = 0,$$

and in particular,

$$\lim_{n \rightarrow \infty} \int_X f_n dx = \int_X f dx.$$

Proof. See e.g. [Rud87, Theorem 1.34].

Q.E.D.

2.6 Fubini's Theorem. *Suppose that f is a measurable function on $X \times Y$ that is also integrable on $X \times Y$, i.e.*

$$\int_{X \times Y} |f(x, y)| d(x, y) < \infty.$$

Then

$$\varphi(x) = \int_Y f(x, y) dy \quad \text{and} \quad \psi(y) = \int_X f(x, y) dx$$

are integrable on X and Y respectively, and

$$\int_{X \times Y} f(x, y) d(x, y) = \int_X \left(\int_Y f(x, y) dy \right) dx = \int_Y \left(\int_X f(x, y) dx \right) dy.$$

Proof. See e.g. [Rud87, Theorem 8.8].

Q.E.D.

The normalized Lebesgue measure is defined to be

$$d\mu_d(x) := (2\pi)^{-\frac{d}{2}} dx,$$

where dx is the standard Lebesgue measure. We use the normalized Lebesgue measure in order to simplify the expressions concerning Fourier transforms.

The Lebesgue spaces $\mathcal{L}^p(\Omega)$ will be normed with respect to the normalized Lebesgue measure, i.e. we set

$$\|f\|_{\mathcal{L}^p(\Omega)} := \begin{cases} \left(\int_{\Omega} |f(x)|^p d\mu_d(x) \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sup_{x \in \Omega} |f(x)|, & p = \infty \end{cases}.$$

We will mostly be interested in $\mathcal{L}^1(\Omega)$ which is the space of all Lebesgue integrable functions on Ω . The space of locally integrable functions on Ω , i.e. functions that are integrable over every compact subset of Ω , will be denoted by $\mathcal{L}_{\text{loc}}^1(\Omega)$.

2.7 Definition. The *convolution* $f * g$ of two functions f and g is defined as

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y)g(y) d\mu_d(y),$$

whenever the choice of f and g guarantees that the integral converges.

Let us also recall some concepts and results of Fourier analysis.

2.8 Definition (The Fourier transform). Let $f \in \mathcal{L}^1$. Then the *Fourier transform* of f is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} d\mu_d(x), \quad \xi \in \mathbb{R}^d. \quad (2.6)$$

The requirement that f is $\mathcal{L}^1$ guarantees that the integral is well-defined. We also recall the following Inversion Theorem.

2.9 Inversion Theorem. Suppose that $f \in \mathcal{L}^1$ and $\widehat{f} \in \mathcal{L}^1$. Then

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\langle x, \xi \rangle} d\mu_d(\xi), \quad (2.7)$$

for almost every $x \in \mathbb{R}^d$.

Proof. See e.g. [Rud87, Theorem 9.11].

Q.E.D.

Remark. Note that if f is continuous then the inversion formula (2.7) holds for every $x \in \mathbb{R}^d$.

As hinted in the introduction, we will find interest in extending the Fourier transform to $\mathbb{C}^d$.

2.10 Definition (Fourier-Laplace transform). The Fourier-Laplace transform Ψ of a function f is defined as

$$\Psi(\zeta) := \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \zeta \rangle} d\mu_d(x),$$

for every $\zeta \in \mathbb{C}^d$ at which the integral is well-defined.

Remark. We denote the Fourier-Laplace transform by Ψ in order to distinguish it from the Fourier transform $\widehat{f}$. In this way we avoid confusion and make certain arguments in our proofs a lot clearer.

2.2. Complex analysis

Many of the concepts and results concerning functions of real variables can analogously be extended to functions of complex variables. In fact, it turns out that these analogues have certain interesting properties that their real counterparts do not possess. Let us first define complex differentiability.

2.11 Definition (Holomorphy and entireness). Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \rightarrow \mathbb{C}$. If $\zeta_0 \in \Omega$, and if

$$f'(\zeta_0) = \lim_{\zeta \rightarrow \zeta_0} \frac{f(\zeta) - f(\zeta_0)}{\zeta - \zeta_0}$$

exists, then $f'(\zeta_0)$ is said to be the complex derivative of f and f is said to be complex differentiable at ζ_0 . If $f'(\zeta_0)$ exists for all $\zeta_0 \in \Omega$ then f is said to be *holomorphic* on Ω . The set of all holomorphic functions on Ω will be denoted by $\mathcal{H}(\Omega)$. If f is holomorphic on the whole complex plane, then f is said to be *entire*.

Similarly as for their real counterparts, the sums, products and compositions of holomorphic functions remain holomorphic.

2.12 Example. Holomorphic (in fact, entire) functions include, but are not limited to: the complex polynomials, the exponential e^ζ , the trigonometric functions $\sin \zeta$ and $\cos \zeta$, the sine cardinal function $\frac{\sin(\zeta)}{\zeta}$, etc. $\triangle$

Let us define a seemingly different class of functions that plays a great role both in real and complex analysis.

2.13 Definition (Analyticity). Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \rightarrow \mathbb{C}$. Then f is said to be *analytic* at $\zeta_0 \in \Omega$ if it is representable by a power series at ζ_0 , in the sense that

$$f(\zeta) = \sum_{j=0}^{\infty} c_j (\zeta - \zeta_0)^j,$$

for some $c_j \in \mathbb{C}$, and where the series converges absolutely in some neighbourhood $\omega \subseteq \Omega$ of ζ_0 . If f is analytic at every $\zeta_0 \in \Omega$ then f is said to be analytic on Ω .

Remark. One can analogously define *real-analytic* functions by taking Ω and ω to be subsets of $\mathbb{R}^d$, and by replacing j in the power series by a multi-index α . Since we will only ever discuss real-analytic functions in the following chapters, we denote by $\mathcal{A}(\Omega)$ the set of all real-analytic functions on $\Omega \subseteq \mathbb{R}^d$.

The coefficients from the power series expansion can be shown to satisfy

$$c_j = \frac{f^{(j)}(\zeta_0)}{j!}.$$

A fundamental result of complex analysis shows that the preceding two definitions define precisely the same class of functions.

2.14 Proposition. Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \rightarrow \mathbb{C}$. Then f is holomorphic on Ω if and only if it is analytic on Ω .

Every function $f : \mathbb{C} \rightarrow \mathbb{C}$ can be considered as a mapping from $\mathbb{R}^2$ to $\mathbb{C}$ by writing $f(\xi + i\eta) = f(\xi, \eta)$. A natural question that therefore arises is how holomorphy relates to $\mathbb{R}^2$ differentiability, in the sense of Definition 2.1. These two concepts are connected via the *Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar{\zeta}} := \frac{1}{2} \left(\frac{\partial}{\partial \xi} + i \frac{\partial}{\partial \eta} \right),$$

as the following proposition suggests.

2.15 Proposition. Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \rightarrow \mathbb{C}$ that is differentiable on Ω as an $\mathbb{R}^2$ map is holomorphic on Ω if and only if the Cauchy-Riemann operator annihilates it, that is,

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = 0$$

for every $\zeta \in \Omega$.

Proof. See e.g. [Rud87, Theorem 11.2].

Q.E.D.

One can even define complex integration similarly as in the case of $\mathbb{R}^2$. Let us first recall the notion of a simply connected domain, and that of a contour.

2.16 Definition. An open set $\Omega \subseteq \mathbb{C}$ is said to be *simply connected* if for every polygon $\Gamma \subset \Omega$, the interior of Γ is also fully contained in Ω .

2.17 Definition. A $\mathcal{C}^1$ -curve is a curve parameterized by a continuously differentiable map $[a, b] \ni t \mapsto \zeta(t) \in \mathbb{C}$ which is injective (except perhaps at the end points $t = a$ and $t = b$). A *contour* is a piecewise $\mathcal{C}^1$ -curve, in the sense that it can be written as a union of finitely many $\mathcal{C}^1$ -curves. A contour is said to be *closed* if its starting and its ending point coincide.

2.18 Definition. Let $\Gamma \subseteq \mathbb{C}$ be a $\mathcal{C}^1$ -curve parameterized by $\zeta : [a, b] \rightarrow \mathbb{C}$ and let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function. Then

$$\int_{\Gamma} f(\zeta) d\zeta := \int_a^b f(\zeta(t)) \zeta'(t) dt.$$

It is remarkable that holomorphic functions are completely characterized by the values of their integrals over closed contours.

2.19 Cauchy's Theorem. Let $\Omega \subseteq \mathbb{C}$ be open and simply connected, suppose that $f \in \mathcal{H}(\Omega)$, and let $\Gamma \subseteq \Omega$ be an arbitrary closed contour. Then

$$\oint_{\Gamma} f(\zeta) d\zeta = 0.$$

Proof. See e.g. [Rud87, Theorems 10.14 and 10.35].

Q.E.D.

2.20 Morera's Theorem. Let $\Omega \subseteq \mathbb{C}$ be open and simply connected, and suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous on Ω . If

$$\oint_{\Gamma} f(\zeta) d\zeta = 0$$

for every closed contour $\Gamma \subset \Omega$, then $f \in \mathcal{H}(\Omega)$.

Proof. For an equivalent result see e.g. [Rud87, Theorem 10.17].

Q.E.D.

A part of this work will concern functions defined by infinite products. The conditions for the holomorphy of such functions will often be of our interest. The following proposition gives one such condition.

2.21 Proposition. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of entire functions such that

$$\sum_{k=1}^{\infty} |1 - f_k(\zeta)|$$

converges uniformly on compact subsets of $\mathbb{C}$. Then the product

$$\prod_{k=1}^{\infty} f_k(\zeta)$$

converges uniformly on compact subsets of $\mathbb{C}$ and forms an entire function.

Proof. See e.g. [Rud87, Theorem 15.6].

Q.E.D.

Another slightly more advanced result will be used in one of our proofs, namely a version of the Phragmén-Lindelöf theorem which we state below.

2.22 Phragmén-Lindelöf Theorem. Let $f \in \mathcal{H}(\mathbb{C}_+) \cap \mathcal{C}(\overline{\mathbb{C}_+})$. If

$$|f(\xi)| \leq C, \quad \xi \in \mathbb{R},$$

for some constant $C > 0$ and, for every $\epsilon > 0$,

$$|f(\zeta)| \leq C_{\epsilon} e^{\epsilon |\zeta|}, \quad \zeta \in \mathbb{C}_+,$$

for some constants $C_{\epsilon} > 0$ possibly dependent on ϵ , then

$$|f(\zeta)| \leq C, \quad \zeta \in \overline{\mathbb{C}_+}.$$

Proof. For a more general result, see e.g. [Tit39, §5.62] or [Con78, §4.4].

Q.E.D.

Most of this work will more generally concern holomorphic functions on $\mathbb{C}^d$. Let us end this section by defining this generalization and recalling a uniqueness result for entire functions.

2.23 Definition. A function $f : \mathbb{C}^d \rightarrow \mathbb{C}$ is said to be holomorphic whenever it is holomorphic in each variable separately.

2.24 Proposition (Uniqueness). Let $f : \mathbb{C}^d \rightarrow \mathbb{C}$ be an entire function that vanishes on $\mathbb{R}^d$. Then f vanishes identically on $\mathbb{C}^d$.

Proof. See e.g. [Rud74, Lemma 7.21].

Q.E.D.

2.3. Distribution theory

It is often unfortunate that not every function is differentiable. However, the notion of differentiability can be extended to a larger class of objects, the so called distributions. Integrating by parts, we find that

$$\int_{\Omega} (\partial^\alpha f) g \, d\mu_d = (-1)^{|\alpha|} \int_{\Omega} f (\partial^\alpha g) \, d\mu_d, \quad (2.8)$$

whenever one of the functions is compactly supported. Notice that f does not need to be smooth in order for the right-hand side of the formula to be well-defined. This motivates us to consider mappings of the form

$$\phi \mapsto \int_{\Omega} f \phi \, d\mu_d, \quad (2.9)$$

where f is locally integrable on Ω , and ϕ is a smooth compactly supported function, also referred to as a test function. The integral is then well-defined due to the assumptions, and it can also be shown that this map defines the function f a.e. (almost everywhere), that is, $f = g$ a.e. if and only if

$$\int_{\Omega} f \phi \, d\mu_d = \int_{\Omega} g \phi \, d\mu_d,$$

for every $\phi \in \mathcal{C}_c^\infty(\Omega)$ [Hör90, Theorems 1.2.4 and 1.2.5]. We may therefore identify every locally integrable function f with the map in (2.9) and in that way define operations such as differentiation of functions which are not differentiable in the usual sense.

In order to generalize these maps, we need to introduce the notion of convergence in $\mathcal{C}^\infty$ and $\mathcal{C}_c^\infty$.

2.25 Definition. Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $\{\phi_j\}_{j=1}^\infty$ be a sequence in $\mathcal{C}^\infty(\Omega)$ and let $\phi \in \mathcal{C}^\infty(\Omega)$. Then we say that $\phi_j \rightarrow \phi$ in $\mathcal{C}^\infty(\Omega)$ as $j \rightarrow \infty$ whenever for every fixed $\alpha \in \mathbb{N}^d$ and every compact set $K \subseteq \Omega$, we have

$$\sup_{x \in K} |\partial^\alpha \phi_j(x) - \partial^\alpha \phi(x)| \rightarrow 0, \quad \text{as } j \rightarrow \infty. \quad (2.10)$$

We say that $\phi_j \rightarrow \phi$ in $\mathcal{C}_c^\infty(\Omega)$ as $j \rightarrow \infty$ whenever for every fixed $\alpha \in \mathbb{N}^d$ we can find a compact set $K \subseteq \Omega$ that contains the supports of ϕ and all ϕ_j , and such that (2.10) holds.

2.26 Definition. ($\mathcal{D}'$ and $\mathcal{E}'$) Let $\Omega \subseteq \mathbb{R}^d$ be open.

- (i) The space $\mathcal{D}'(\Omega)$ is the set of all linear forms $u : \mathcal{C}_c^\infty(\Omega) \rightarrow \mathbb{C}$ that are continuous in the sense that $u(\phi_j) \rightarrow u(\phi)$ whenever $\phi_j \rightarrow \phi$ in $\mathcal{C}_c^\infty(\Omega)$ as $j \rightarrow \infty$.

- (ii) The space $\mathcal{E}'(\Omega)$ is the set of all linear forms $u : \mathcal{C}^\infty(\Omega) \rightarrow \mathbb{C}$ that are continuous in the sense that $u(\phi_j) \rightarrow u(\phi)$ whenever $\phi_j \rightarrow \phi$ in $\mathcal{C}^\infty(\Omega)$ as $j \rightarrow \infty$.

The elements of $\mathcal{D}'(\Omega)$ are called *distributions*. The space $\mathcal{E}'(\Omega)$ can be identified with the distributions whose support is compact, as we will see later. The continuity condition for $\mathcal{D}'(\Omega)$ can also be restated in the following way.

2.27 Proposition. *A linear form $u : \mathcal{C}_c^\infty(\Omega) \rightarrow \mathbb{C}$ defines an element in $\mathcal{D}'(\Omega)$ if and only if for every compact set $K \subseteq \Omega$ there exist constants $C = C_K > 0$ and $N > 0$ such that*

$$|u(\phi)| \leq C \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \phi(x)|, \quad (2.11)$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$ with $\text{supp}(\phi) \subseteq K$.

We will hereafter assume that the domain space Ω is the whole $\mathbb{R}^d$.

2.28 Example. As already hinted at the start of this section, every $f \in \mathcal{L}_{\text{loc}}^1$ can be identified with the distribution

$$u_f(\phi) = \int_{\mathbb{R}^d} f(x)\phi(x) d\mu_d(x).$$

Whenever we say that a distribution is a function, we mean that there exists a function that gives rise to the distribution in the sense of the formula above. $\triangle$

2.29 Example. One of the simplest distributions which is not a function is the so-called *Dirac delta* δ . It is formally defined as the number

$$\delta(\phi) = \phi(0). \quad \triangle$$

Definition 2.3 of the support of a function also generalizes to distributions in the following way.

2.30 Definition. Let $u \in \mathcal{D}'$ and let $\omega \subseteq \mathbb{R}^d$ be open. Then u is said to vanish on ω if $u(\phi) = 0$ for every $\phi \in \mathcal{C}_c^\infty$ with $\text{supp}(\phi) \subseteq \omega$. The support of u is defined as the complement of the largest open set on which u vanishes.

2.31 Example. The support of the Dirac delta is the singleton $\{0\}$, so that δ is a compactly supported distribution. $\triangle$

By finding a cutoff function $\chi \in \mathcal{C}_c^\infty$ such that $\chi = 1$ on a neighbourhood of the support of u , one can extend every compactly supported distribution u to a functional on $\mathcal{C}^\infty$ in a unique way by means of the formula

$$u(\phi) = u(\chi\phi).$$

Conversely, the restriction of $u \in \mathcal{E}'$ to $\mathcal{C}_c^\infty$ defines a compactly supported distribution. More precisely, it is possible to show the following result.

2.32 Proposition. *The space $\mathcal{E}'$ is identical to the space of elements in $\mathcal{D}'$ whose support is compact.*

Proof. See e.g. [Hör90, Theorem 2.3.1].

Q.E.D.

One therefore usually refers to elements of $\mathcal{E}'$ as compactly supported distributions. Motivated by formulae from integration, we also define the following operations on distributions.

2.33 Definition. Let $u \in \mathcal{D}'$ and $\phi \in \mathcal{C}_c^\infty$, or else $u \in \mathcal{E}'$ and $\phi \in \mathcal{C}^\infty$. Let also $\psi \in \mathcal{C}^\infty$. Then

$$\psi u(\phi) = u(\psi\phi), \quad \partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi), \quad \text{and} \quad (u * \phi)(x) = u(\phi(x - \cdot)).$$

Here and throughout the rest of this work the dot $(\cdot)$ represents the variable with respect to which u acts on the test function.

2.34 Example. The partial derivatives of the Dirac delta are given by

$$\partial^\alpha \delta(\phi) = (-1)^{|\alpha|} \delta(\partial^\alpha \phi) = (-1)^{|\alpha|} \partial^\alpha \phi(0). \quad \triangle$$

Since $u \in \mathcal{E}'$ accepts arguments from $\mathcal{C}^\infty$, and the function $x \mapsto e^{-i\langle x, \xi \rangle}$ is infinitely many times differentiable for every fixed $\xi \in \mathbb{R}^d$, the following definition is natural.

2.35 Definition. The Fourier transform of a compactly supported distribution is the function

$$\widehat{u}(\xi) = u\left(e^{-i\langle \cdot, \xi \rangle}\right).$$

This definition is consistent with our identification in Example 2.28 and the Fourier transform that we have previously defined on $\mathcal{L}^1$.

For a deeper exposition of the subject of distributions, let us refer to [Hör90]. We will also give a more detailed analysis of the results we need once we start discussing Gevrey ultradistributions in the fourth chapter.

3. Gevrey Functions

3.1. The spaces $\mathcal{G}^s$ and $\mathcal{G}_c^s$

The Gevrey spaces were first introduced by M. Gevrey in [Gev18]. In his work, he studied the heat operator on $\mathbb{R}^d$, $d \geq 2$, defined as

$$\frac{\partial}{\partial x_d} - \sum_{j=1}^{d-1} \frac{\partial^2}{\partial x_j^2},$$

and whose fundamental solution is given by

$$E(x) = \begin{cases} (4\pi x_d)^{\frac{1-d}{2}} \exp\left(-\frac{1}{4x_d} \cdot (x_1^2 + \dots + x_{d-1}^2)\right) & , \quad x_d > 0 \\ 0 & , \quad x_d \leq 0 \end{cases}.$$

The function E is not real-analytic on the hyperplane $x_d = 0$. However, it is a standard result in analysis that E is smooth everywhere except at the origin, see e.g. [Hör90, Example 1.1.3 and Lemma 1.2.3], and it is also possible to show [Rod93] that E satisfies for all compact sets $K \subseteq \mathbb{R}^d \setminus \{0\}$, the estimate

$$\sup_{x \in K} |\partial^\alpha E(x)| \lesssim \tau^{|\alpha|} \alpha!^2, \quad \alpha \in \mathbb{N}^d, \quad (3.1)$$

for some constant $\tau = \tau_K > 0$. This estimate is somewhat reminiscent of the Cauchy characterization of real-analytic functions [KP92, Proposition 1.2.10], which states that a function f belongs to $\mathcal{A}$ if and only if it satisfies the estimate

$$\sup_{x \in K} |\partial^\alpha f(x)| \lesssim \tau^{|\alpha|} \alpha!, \quad \alpha \in \mathbb{N}^d, \quad (3.2)$$

for all compact sets $K \subseteq \mathbb{R}^d$ and some constant $\tau = \tau_K > 0$. This suggests that one might find interest in measuring the regularity of a function by investigating the exponent of $\alpha!$ in estimates of type (3.1) and (3.2). In particular, Gevrey introduced the following class of functions.

3.1 Definition ($\mathcal{G}^s$). Let $\Omega \subseteq \mathbb{R}^d$ be open and let $s \geq 1$ be a real number. The *Gevrey class* $\mathcal{G}^s(\Omega)$ of order s on Ω is defined as the space of all functions $f \in \mathcal{C}^\infty(\Omega)$ that satisfy the Gevrey estimate

$$\sup_{x \in K} |\partial^\alpha f(x)| \lesssim \tau^{|\alpha|} \alpha!^s, \quad \alpha \in \mathbb{N}^d,$$

for every compact set $K \subseteq \Omega$ and some constant $\tau = \tau_K > 0$. Such functions are said to be Gevrey functions of order s . The space of Gevrey functions of order s whose support is compact is denoted by $\mathcal{G}_c^s(\Omega)$.

We will hereafter assume that $\Omega = \mathbb{R}^d$. As already hinted, when $s = 1$ the Gevrey space $\mathcal{G}^1$ and the space of real-analytic functions $\mathcal{A}$ are in fact identical. If $1 < s < t < \infty$, we have the strict inclusions [Rod93]

$$\mathcal{A} \subsetneq \mathcal{G}^s \subsetneq \mathcal{G}^t \subsetneq \mathcal{C}^\infty,$$

so that one can think of the classes of Gevrey functions as intermediate spaces between the spaces of real-analytic functions and smooth functions. Let us present the classical example of a Gevrey function.

3.2 Example. Consider for $s > 1$ the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\phi(x) = \begin{cases} \exp\left(-x^{-\frac{1}{s-1}}\right) & , \quad x > 0 \\ 0 & , \quad x \leq 0 \end{cases}.$$

Then $\phi \in \mathcal{G}^s$. For the proof, let us refer to [CC05, Lemma 1]. $\triangle$

The Gevrey class is quite stable, as the following proposition suggests.

3.3 Proposition. Suppose that $s \geq 1$, and let $f, g \in \mathcal{G}^s$ and $a, b \in \mathbb{C}$. Then $af + bg \in \mathcal{G}^s$, $fg \in \mathcal{G}^s$, $\partial^\alpha f \in \mathcal{G}^s$, and $f \circ g \in \mathcal{G}^s$.

Proof. See e.g. [Rod93, Propositions 1.4.5, 1.4.6, and Remark 1.4.7]. *Q.E.D.*

In this work, we are particularly interested in the spaces $\mathcal{G}_c^s$. Note that the space $\mathcal{G}_c^1$ is trivial, since the only real-analytic function with compact support is the zero function. However, the following example illustrates that the spaces $\mathcal{G}_c^s$ are non-trivial when $s > 1$.

3.4 Example. Let ϕ be as in Example 3.2 and consider the function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\psi(x) = C\phi(1 - |x|^2),$$

where C is a normalization constant chosen so that $\int_{\mathbb{R}^d} \psi d\mu_d = 1$. It follows that ψ is a Gevrey function of order s in view of Proposition 3.3 since $1 - |x|^2 \in \mathcal{A} \subseteq \mathcal{G}^s$. Furthermore, ψ is supported in the unit ball $\bar{B}_1$ by construction, so that $\psi \in \mathcal{G}_c^s$. $\triangle$

Let us present another three examples of functions in $\mathcal{G}_c^s$ which will become useful to us once we start discussing ultradistributions.

3.5 Example (Approximate identity). Let ψ be as in Example 3.4 and define for $\epsilon > 0$ the functions $\psi_\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$\psi_\epsilon(x) = \epsilon^{-d} \psi(\epsilon^{-1}x).$$

It follows by construction that each ψ_ϵ belongs to $\mathcal{G}_c^s$ with support in the ball $\bar{B}_\epsilon$, and that $\int_{\mathbb{R}^d} \psi_\epsilon d\mu_d = 1$. The family $\{\psi_\epsilon\}_{\epsilon>0}$ of such functions is usually referred to as an *approximate identity*.

Let us also note that, for every $\xi \in \mathbb{R}^d$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \widehat{\psi}_\epsilon(\xi) &= \lim_{\epsilon \rightarrow 0^+} \epsilon^{-d} \int_{\bar{B}_\epsilon} \psi(\epsilon^{-1}x) e^{-i\langle x, \xi \rangle} d\mu_d(x) = \lim_{\epsilon \rightarrow 0^+} \int_{\bar{B}_1} \psi(x) e^{-i\epsilon \langle x, \xi \rangle} d\mu_d(x) \\ &= \int_{\bar{B}_1} \psi(x) \lim_{\epsilon \rightarrow 0^+} e^{-i\epsilon \langle x, \xi \rangle} d\mu_d(x) = \int_{\bar{B}_1} \psi(x) d\mu_d(x) = 1, \end{aligned}$$

where the interchange of the limit and the integral is allowed by the Dominated Convergence Theorem since the integration is taken over a compact set. Hence, $\widehat{\psi}_\epsilon$ converges pointwise to the identity function as $\epsilon \rightarrow 0^+$. $\triangle$

3.6 Example (Cutoff function). Let $K \subseteq \mathbb{R}^d$ be compact, let ψ and ψ_ϵ be as in Examples 3.4 and 3.5, and let ν be the characteristic function of $K_{2\epsilon/3} = \{x + y : x \in K, |y| \leq \frac{2\epsilon}{3}\}$. Define for $\epsilon > 0$ the function $\chi_\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\chi_\epsilon(x) = (\nu * \psi_{\epsilon/3})(x).$$

Then χ_ϵ is a *cutoff function* of Gevrey class, since $\chi_\epsilon = 1$ on a neighbourhood of K (more precisely, on $K_{\epsilon/3}$), and $\chi_\epsilon \in \mathcal{G}_c^s$ with $\text{supp}(\chi_\epsilon) = K_\epsilon$. We also observe that, for every $x \in K_\epsilon$,

$$|\partial^\alpha \chi_\epsilon(x)| \leq \left(\frac{\epsilon}{3}\right)^{-|\alpha|} \int_{\mathbb{R}^d} |\partial^\alpha \psi(y)| d\mu_d(y) \lesssim \left(\frac{\epsilon}{3}\right)^{-|\alpha|} \tau_\epsilon^{|\alpha|} \alpha!^s = \tau_\epsilon^{|\alpha|} \alpha!^s,$$

where the second inequality follows since $\psi \in \mathcal{G}_c^s$, and in the final step we put $\tau_\epsilon = \frac{3\tau}{\epsilon}$. Here we want to emphasize that the constant τ_ϵ depends on ϵ . This will cause some difficulties in the next chapter. The proof of these claims is analogous to the classical case, so let us refer to [Hör90, Theorem 1.4.1]. $\triangle$

3.7 Example (Partition of unity). Let $K \subseteq \mathbb{R}^d$ be compact, and let $\{\Omega_j\}_{j=1}^n$ be a finite collection of open sets that cover K . Choose compact subsets $K_j \subseteq \Omega_j$ so that the collection $\{K_j\}_{j=1}^n$ still covers K . Let $\chi_j \in \mathcal{G}_c^s$ be cutoff functions such that $\text{supp}(\chi_j) \subseteq \Omega_j$ and $\chi_j = 1$ on a neighbourhood of K_j . Put

$$\phi_1 = \chi_1, \quad \phi_2 = \chi_2(1 - \chi_1), \quad \dots, \quad \phi_n = \chi_n(1 - \chi_1) \cdots (1 - \chi_{n-1}).$$

Then the collection $\{\phi_j\}_{j=1}^n$ is a *partition of unity* of Gevrey class, in that

$$\sum_{j=1}^n \phi_j = 1 \quad \text{on a neighbourhood of } K.$$

Observe that this also allows us to decompose every $\varphi \in \mathcal{G}_c^s$ with $\text{supp}(\varphi) \subseteq K$ into a sum of Gevrey cutoff functions with support contained in open sets that cover K . This can be done by multiplying each ϕ_j above by φ , and this product still lies in $\mathcal{G}_c^s$ due to Proposition 3.3. Also note that we can always choose Ω_j to be convex sets.

The proof that all of this is correct is a simple modification of its classical counterpart, see [Hör90, Theorem 1.4.4 & 1.4.5]. $\triangle$

3.2. The Paley-Wiener Theorem for $\mathcal{G}_c^s$

It is quite remarkable that the Paley-Wiener Theorem for $\mathcal{C}_c^\infty$ can be sharpened to obtain its analogue for $\mathcal{G}_c^s$. The difference is that the polynomial decay of the Fourier-Laplace transform is replaced by an exponential one. Let us state this more precisely.

3.8 Paley-Wiener Theorem for $\mathcal{G}_c^s$. *Let $K \subseteq \mathbb{R}^d$ be a compact convex set. A function f belongs to $\mathcal{G}_c^s$ and has its support contained in K if and only if its Fourier-Laplace transform Ψ is an entire function satisfying the estimate*

$$|\Psi(\zeta)| \lesssim \exp \left(-\lambda |\zeta|^{\frac{1}{s}} + H_K(\text{Im } \zeta) \right), \quad (3.3)$$

for every $\zeta \in \mathbb{C}^d$, and some constant $\lambda > 0$.

We recall that the supporting function $H_K : \mathbb{R}^d \rightarrow \mathbb{R}$ of a compact convex set K is defined as

$$H_K(\eta) := \sup_{t \in K} \langle t, \eta \rangle.$$

Before continuing with the proof of the theorem, let us state a result which will be useful to us on a few occasions throughout the thesis.

3.9 Lemma. *Let $s > 1$ be fixed. Then for every $\lambda > 0$ there exist constants $C_\lambda > 0$ and $M > 0$ such that*

$$\sup_{k \in \mathbb{N}} \left\{ \frac{\lambda^k}{k!} |\zeta|^{\frac{k}{s}} \right\} \leq C_\lambda \sup_{\alpha \in \mathbb{N}^d} \left\{ \left(\frac{M\lambda}{|\alpha|} \right)^{s|\alpha|} |\zeta^\alpha| \right\},$$

for every $\zeta \in \mathbb{C}^d$.

Proof. First suppose that $|\zeta| \geq 1$ and let m be the least integer greater or equal to s , so that $s \leq m \leq s+1$. Given $k \in \mathbb{N}$, find $n \in \mathbb{N}$ such that $\frac{k}{s} \leq n \leq \frac{k}{s} + 1$. Then

$$k \leq sn \leq k + s \leq k + m, \quad (3.4)$$

and since k and m are integers, we have

$$(k+m)! \leq 2^{k+m} k! m! \lesssim 2^k k! \leq 2^{sn} k!, \quad (3.5)$$

where the second inequality follows since $2^m m!$ is bounded by a constant that depends only on s , and the final inequality is due to (3.4). Furthermore, if $n \geq 1$, then the Taylor series expansion of e^n combined with (3.4) shows that

$$(k+m)! \geq e^{-n} n^{k+m} \geq e^{-n} n^{sn},$$

whereas if $n = 0$ then $(k+m)! \geq e^{-n} n^{sn}$ follows trivially. A combination of this estimate and (3.5) implies that

$$\frac{\lambda^k}{k!} \lesssim C_\lambda \frac{2^{sn} \lambda^{sn}}{(k+m)!} \leq C_\lambda \frac{2^{sn} \lambda^{sn}}{e^{-n} n^{sn}} = C_\lambda \left(\frac{M_0 \lambda}{n} \right)^{sn},$$

with a suitably chosen constant $M_0 > 0$, and where the first inequality follows regardless of whether $\lambda \geq 1$ or $\lambda < 1$, since in the former case the inequality is immediate with $C_\lambda = 1$, and in the latter case we may write $\lambda^k = \lambda^{-s} \lambda^{k+s} \leq \lambda^{-s} \lambda^{sn} = C_\lambda \lambda^{sn}$. Hence,

$$\begin{aligned} \frac{\lambda^k}{k!} |\zeta|^{\frac{k}{s}} &\lesssim C_\lambda \left(\frac{M_0 \lambda}{n} \right)^{sn} |\zeta|^n \leq C_\lambda \left(\frac{M \lambda}{n} \right)^{sn} \max_{|\alpha|=n} |\zeta^\alpha| \\ &= C_\lambda \max_{|\alpha|=n} \left\{ \left(\frac{M \lambda}{|\alpha|} \right)^{s|\alpha|} |\zeta^\alpha| \right\} \leq C_\lambda \sup_{\alpha \in \mathbb{N}^d} \left\{ \left(\frac{M \lambda}{|\alpha|} \right)^{s|\alpha|} |\zeta^\alpha| \right\} \end{aligned} \quad (3.6)$$

where in the second inequality we choose M accordingly by taking into account the estimate $|\zeta|^n \leq d^n \max_{|\alpha|=n} |\zeta^\alpha|$. The result follows by taking the supremum over k .

The case when $|\zeta| \leq 1$ is treated analogously by finding $n \in \mathbb{N}$ such that $n \leq \frac{k}{s}$.
Q.E.D.

Proof of Theorem 3.8. We put $\xi = \operatorname{Re}(\zeta)$ and $\eta = \operatorname{Im}(\zeta)$ throughout the proof as usual.

Let us begin by proving the necessity part. Let $f \in \mathcal{G}_c^s$ with $\operatorname{supp}(f) \subseteq K$, and let Ψ be its Fourier-Laplace transform, i.e.

$$\Psi(\zeta) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \zeta \rangle} d\mu_d(x), \quad \zeta \in \mathbb{C}^d,$$

where the integral is well-defined. In fact, for $x \in K$ and $\zeta \in \mathbb{C}^d$, we have

$$\left| e^{-i\langle x, \zeta \rangle} \right| = \left| e^{-i\langle x, \xi \rangle} e^{\langle x, \eta \rangle} \right| = e^{\langle x, \eta \rangle} \leq \exp \left(\sup_{t \in K} \langle t, \eta \rangle \right) = e^{H_K(\eta)},$$

which gives

$$\begin{aligned} |\Psi(\zeta)| &= \left| \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \zeta \rangle} d\mu_d(x) \right| = \left| \int_K f(x) e^{-i\langle x, \zeta \rangle} d\mu_d(x) \right| \\ &\leq \int_K |f(x)| \left| e^{-i\langle x, \zeta \rangle} \right| d\mu_d(x) \leq e^{H_K(\eta)} \|f\|_{\mathcal{L}^1}, \end{aligned}$$

and since $f \in \mathcal{L}^1$, Ψ is well-defined on $\mathbb{C}^d$.

To show that Ψ is continuous, let $\{\zeta_k\}_{k=1}^\infty \subset \mathbb{C}$ be a sequence converging to $\zeta \in \mathbb{C}$. Then

$$|\Psi(\zeta) - \Psi(\zeta_k)| \leq \int_K |f(x)| \left| e^{-i\langle x, \zeta \rangle} - e^{-i\langle x, \zeta_k \rangle} \right| d\mu_d(x).$$

Since the integral is taken over a compact set and the integrand is well-behaved, the Dominated Convergence Theorem may be applied directly so that letting $k \rightarrow \infty$ inside the integral gives continuity.

To show that Ψ is entire, let $\Gamma \subseteq \mathbb{C}$ be an arbitrary closed contour. Then, for all $1 \leq j \leq d$,

$$\begin{aligned} \oint_{\Gamma} \Psi(\zeta) d\zeta_j &= \oint_{\Gamma} \int_K f(x) e^{-ix\zeta} d\mu_d(x) d\zeta_j \\ &= \int_K f(x) \oint_{\Gamma} e^{-ix\zeta} d\zeta_j d\mu_d(x) = 0, \end{aligned}$$

where the second equality follows by Fubini's Theorem since the integrand is $\mathcal{L}^1(K \times \Gamma)$, and the final equality is due to Cauchy's Theorem since the complex exponential is entire. An application of Morera's Theorem therefore shows that Ψ is an entire function.

Finally, we must prove that Ψ satisfies the estimate (3.3). Note first that, for any multi-index $\alpha \in \mathbb{N}^d$, integration by parts gives

$$\zeta^\alpha \Psi(\zeta) = (-i)^{|\alpha|} \int_K (\partial^\alpha f(x)) e^{-i\langle x, \zeta \rangle} d\mu_d(x), \quad (3.7)$$

where the boundary terms of the integrations by parts vanish since f is compactly supported. Hence,

$$\begin{aligned} |\zeta^\alpha \Psi(\zeta)| &\leq \int_K |\partial^\alpha f(x)| \left| e^{-i\langle x, \zeta \rangle} \right| d\mu_d(x) \lesssim \sup_{x \in K} \left(|\partial^\alpha f(x)| e^{\langle x, \eta \rangle} \right) \\ &\lesssim \tau^{|\alpha|} \alpha!^s e^{H_K(\eta)} \leq \tau^{|\alpha|} |\alpha|^{s|\alpha|} e^{H_K(\eta)}, \end{aligned}$$

for some constant $\tau > 0$. Lemma 3.9 therefore shows that for every $\lambda > 0$ there exist constants $C_\lambda, M > 0$ such that

$$\begin{aligned} \sup_{k \in \mathbb{N}} \left\{ \frac{(2\lambda)^k}{k!} |\zeta|^{\frac{k}{s}} |\Psi(\zeta)| \right\} &\leq C_\lambda \sup_{\alpha \in \mathbb{N}^d} \left\{ \left(\frac{2M\lambda}{|\alpha|} \right)^{s|\alpha|} |\zeta^\alpha \Psi(\zeta)| \right\} \\ &\lesssim C_\lambda \sup_{\alpha \in \mathbb{N}^d} \left(2\tau^{\frac{1}{s}} M\lambda \right)^{s|\alpha|} e^{H_K(\eta)}. \end{aligned}$$

Hence,

$$\begin{aligned} \exp\left(\lambda|\zeta|^{\frac{1}{s}}\right)|\Psi(\zeta)| &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |\zeta|^{\frac{k}{s}} |\Psi(\zeta)| \leq \sup_{k \in \mathbb{N}} \left\{ \frac{(2\lambda)^k}{k!} |\zeta|^{\frac{k}{s}} |\Psi(\zeta)| \right\} \sum_{k=0}^{\infty} 2^{-k} \\ &\lesssim C_\lambda \sup_{\alpha \in \mathbb{N}^d} \left(2\tau^{\frac{1}{s}} M \lambda \right)^{s|\alpha|} e^{H_K(\eta)} \lesssim e^{H_K(\eta)}, \end{aligned}$$

where in the final step we fix $\lambda = (2\tau^{\frac{1}{s}} M)^{-1}$. Thus,

$$|\Psi(\zeta)| \lesssim \exp\left(-\lambda|\zeta|^{\frac{1}{s}} + H_K(\eta)\right),$$

for every $\zeta \in \mathbb{C}^d$, and some $\lambda > 0$.

To prove sufficiency, suppose that Ψ is an entire function that satisfies the estimate (3.3). Let f be the inverse Fourier transform of Ψ , i.e.

$$f(x) = \int_{\mathbb{R}^d} \Psi(\xi) e^{i\langle x, \xi \rangle} d\mu_d(\xi), \quad \xi \in \mathbb{R}^d, \quad (3.8)$$

where the integral is well-defined, since by (3.3), combined with the estimate $1 + \frac{t^n}{n!} \leq e^t$, we have

$$\begin{aligned} |f(x)| &\leq \int_{\mathbb{R}^d} |\Psi(\xi)| d\mu_d(\xi) \lesssim \int_{\mathbb{R}^d} \exp\left(-\lambda|\xi|^{\frac{1}{s}}\right) d\mu_d(\xi) \\ &\leq \int_{\mathbb{R}^d} \left(1 + \frac{\lambda^n |\xi|^{\frac{n}{s}}}{n!}\right)^{-1} d\mu_d(\xi) < \infty, \end{aligned} \quad (3.9)$$

given that n is chosen large enough.

The continuity of f follows in a similar manner as in the first part of the proof, where the difference is that we use (3.9) instead of the compact support to motivate the use of the Dominated Convergence Theorem. Furthermore, by standard arguments (see e.g. [Rud74, Theorem 7.4]) we find that, for any multi-index $\alpha \in \mathbb{N}^d$,

$$\partial^\alpha f(x) = i^{|\alpha|} \int_{\mathbb{R}^d} \xi^\alpha \Psi(\xi) e^{i\langle x, \xi \rangle} d\mu_d(\xi)$$

where the integral is well-defined since $\xi^\alpha \Psi(\xi) \in \mathcal{L}^1$ by similar arguments as in (3.9). It follows that $f \in \mathcal{C}^\infty$.

Next, we must show that f satisfies the Gevrey estimate. By the arguments above, we have

$$\begin{aligned} |\partial^\alpha f(x)| &\leq \int_{\mathbb{R}^d} |\xi^\alpha \Psi(\xi)| d\mu_d(\xi) \\ &\leq \int_{\mathbb{R}^d} |\xi^\alpha| \exp\left(-\lambda|\xi|^{\frac{1}{s}}\right) d\mu_d(\xi) \\ &= \int_{\mathbb{R}^d} |\xi^\alpha| \exp\left(-\frac{\lambda}{2}|\xi|^{\frac{1}{s}}\right) \exp\left(-\frac{\lambda}{2}|\xi|^{\frac{1}{s}}\right) d\mu_d(\xi). \end{aligned} \quad (3.10)$$

Using the fact that $t^{sn}e^{-st} \leq n!^s$, we get with $n = |\alpha|$ and $t = \frac{\lambda}{2s} |\xi|^{\frac{1}{s}}$ that

$$\begin{aligned} |\xi^\alpha| \exp\left(-\frac{\lambda}{2} |\xi|^{\frac{1}{s}}\right) &\leq |\xi|^{|\alpha|} \exp\left(-\frac{\lambda}{2} |\xi|^{\frac{1}{s}}\right) \\ &= \left(\frac{2s}{\lambda}\right)^{s|\alpha|} \left(\frac{\lambda}{2s} |\xi|^{\frac{1}{s}}\right)^{s|\alpha|} \exp\left(-\frac{\lambda}{2} |\xi|^{\frac{1}{s}}\right) \\ &\leq \left(\frac{2s}{\lambda}\right)^{s|\alpha|} |\alpha|!^s \leq \tau^{|\alpha|} \alpha!^s, \end{aligned} \quad (3.11)$$

where τ is a constant chosen suitably by taking into account that $|\alpha|! \leq d^{|\alpha|} \alpha!$. Combining (3.10) and (3.11) we obtain

$$|\partial^\alpha f(x)| \leq \tau^{|\alpha|} \alpha!^s \int_{\mathbb{R}^d} \exp\left(-\frac{\lambda}{2} |\xi|^{\frac{1}{s}}\right) d\mu_d(\xi) \lesssim \tau^{|\alpha|} \alpha!^s, \quad x \in \mathbb{R}^d,$$

where the final inequality follows since the integral is finite in view of (3.9). This proves that $f \in \mathcal{G}^s$.

To show that f is compactly supported, we first claim that the integral in (3.8) remains invariant under the variable change $\xi \mapsto \xi + i\eta = \zeta$ for fixed $\eta \in \mathbb{R}^d$, i.e. that

$$\int_{\mathbb{R}^d} \Psi(\xi) e^{i\langle x, \xi \rangle} d\mu_d(\xi) = \int_{\mathbb{R}^d} \Psi(\xi + i\eta) e^{i\langle x, \xi + i\eta \rangle} d\mu_d(\xi). \quad (3.12)$$

Note that it suffices to show that this holds in each variable separately, since the integrand is $\mathcal{L}^1$, so that Fubini's Theorem allows us to interchange the order of integration. Fix j , $1 \leq j \leq d$. To simplify our arguments, we temporarily introduce the notation

$$\zeta'(z) = (\zeta_1, \dots, z, \dots, \zeta_d),$$

for $z \in \mathbb{C}$, where z replaces the j -th coordinate of ζ , and all the other coordinates of ζ are fixed. Hence, our problem reduces to showing

$$\int_{\mathbb{R}} \Psi(\zeta'(\xi_j)) e^{i\langle x, \zeta'(\xi_j) \rangle} d\mu_1(\xi_j) = \int_{\mathbb{R}} \Psi(\zeta'(\xi_j + i\eta_j)) e^{i\langle x, \zeta'(\xi_j + i\eta_j) \rangle} d\mu_1(\xi_j). \quad (3.13)$$

Let $\rho \in \mathbb{R}_+$, and let $\Gamma_\rho \subseteq \mathbb{C}$ be the positively oriented rectangular contour with vertices in $\pm\rho + i0$ and $\pm\rho + i\eta_j$. Note that η_j is fixed since η is, and we may without loss of generality assume that $\eta_j > 0$. The contour is composed of four straight line segments, which we denote by $\Gamma_{k,\rho}$, $k = 1, \dots, 4$, as shown in Figure 3.2.

By Cauchy's Theorem, we have that

$$\oint_{\Gamma_\rho} \Psi(\zeta'(z)) e^{i\langle x, \zeta'(z) \rangle} dz = 0, \quad (3.14)$$

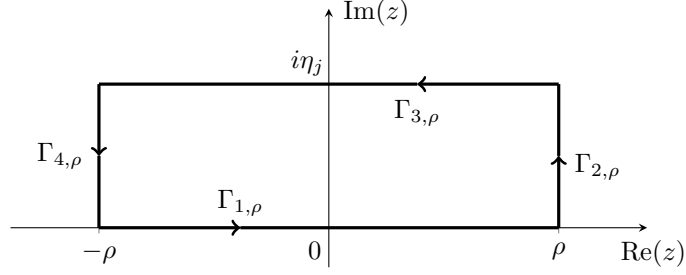


Figure 3.1.: The contour Γ_ρ

for all ρ , since the integrand is a product of entire functions. Now we claim that the integrals over the vertical edges of Γ_ρ approach zero as $\rho \rightarrow \infty$. In fact, this follows almost directly from the estimate (3.3). Since $e^{-t} \leq n! t^{-n}$, we find that

$$\begin{aligned} |\Psi(\zeta'(z))| &\lesssim \exp\left(-\lambda |\zeta'(z)|^{\frac{1}{s}} + H_K(\text{Im } \zeta'(z))\right) \\ &\leq n! \lambda^{-n} |\zeta'(z)|^{-\frac{n}{s}} e^{H_K(\text{Im } \zeta'(z))} \\ &\lesssim |\text{Re}(z)|^{-\frac{n}{s}} e^{H_K(\text{Im } \zeta'(z))}. \end{aligned} \quad (3.15)$$

where the final inequality follows since we just decrease the expression inside the modulus. Furthermore,

$$\left| e^{i\langle x, \zeta'(z) \rangle} \right| = e^{-\langle x, \text{Im}(\zeta'(z)) \rangle} \leq e^{|x| |\text{Im}(\zeta'(z))|}. \quad (3.16)$$

For all $z \in \Gamma_{2,\rho}$, we have $|\text{Re}(z)| = \rho$, and $|\text{Im}(z)| \leq |\eta_j|$, so that $|\text{Im}(\zeta'(z))| \leq |\eta|$, and thus by (3.15) and (3.16),

$$\left| \int_{\Gamma_{2,\rho}} \Psi(\zeta'(z)) e^{i\langle x, \zeta'(z) \rangle} dz \right| \lesssim \int_{\Gamma_{2,\rho}} |\rho|^{-\frac{n}{s}} e^{H_K(\eta)} e^{|x| |\eta|} |dz| \lesssim |\rho|^{-\frac{n}{s}},$$

where the last inequality follows since the exponentials inside the integral are just fixed constants, and so is the arc length of $\Gamma_{2,\rho}$. It follows that the integral over $\Gamma_{2,\rho}$ approaches zero as $\rho \rightarrow \infty$. An equivalent argument shows that the same is true for $\Gamma_{4,\rho}$. This fact, combined with (3.14), shows that the sum of the integrals over the horizontal edges must be equal to 0 when $\rho \rightarrow \infty$, i.e.

$$\lim_{\rho \rightarrow \infty} \int_{\Gamma_{1,\rho}} \Psi(\zeta'(z)) e^{i\langle x, \zeta'(z) \rangle} dz + \lim_{\rho \rightarrow \infty} \int_{\Gamma_{3,\rho}} \Psi(\zeta'(z)) e^{i\langle x, \zeta'(z) \rangle} dz = 0,$$

which is precisely equivalent to (3.13). Hence, we conclude that

$$f(x) = \int_{\mathbb{R}^d} \Psi(\zeta) e^{i\langle x, \zeta \rangle} d\mu_d(\xi), \quad (3.17)$$

for all $\zeta = \xi + i\eta \in \mathbb{R}^d$, i.e. f is independent of the choice of η .

Now we find using the assumed estimate (3.3) that

$$\begin{aligned} |f(x)| &\leq \int_{\mathbb{R}^d} |\Psi(\zeta)| e^{-\langle x, \eta \rangle} d\mu_d(\xi) \\ &\lesssim \int_{\mathbb{R}^d} \exp\left(-\lambda |\zeta|^{\frac{1}{s}} + H_K(\eta) - \langle x, \eta \rangle\right) d\mu_d(\xi) \\ &\lesssim \exp(H_K(\eta) - \langle x, \eta \rangle) \int_{\mathbb{R}^d} \exp\left(-\lambda |\xi|^{\frac{1}{s}}\right) d\mu_d(\xi), \end{aligned} \quad (3.18)$$

where the integral converges as we have seen in (3.9). Let $x \notin K$. Then K and $\{x\}$ are disjoint compact convex sets, so that Theorem 2.4 implies the existence of a unit vector $v \in \mathbb{R}^d$ such that $\sup_{t \in K} \langle t, v \rangle - \langle x, v \rangle < 0$. If we put $\eta = \gamma v$ for $\gamma \in \mathbb{R}_+$, the estimate (3.18) gives

$$|f(x)| \lesssim \exp(H_K(\eta) - \langle x, \eta \rangle) = \exp(\gamma(H_K(v) - \langle x, v \rangle)).$$

Letting $\gamma \rightarrow \infty$, we find that $f(x) = 0$ for $x \notin K$. In particular, this shows that f has its support contained in K , so that $f \in \mathcal{G}_c^s$.

Finally, we must show that Ψ truly is the Fourier-Laplace transform of f . Since $f \in \mathcal{G}_c^s \subseteq \mathcal{L}^1$, and $\Psi \in \mathcal{L}^1$ in view of (3.9), the Inversion Theorem combined with (3.8) implies that $\Psi(\xi) = \hat{f}(\xi)$ for $\xi \in \mathbb{R}^d$. But then Ψ and the Fourier-Laplace transform of f are entire functions that coincide on $\mathbb{R}^d$, so that they must coincide on the whole $\mathbb{C}^d$ by Proposition 2.24.

Q.E.D.

We end this section by presenting an interesting example of a function that belongs to $\mathcal{G}_c^s$ as a consequence the Paley-Wiener Theorem.

3.10 Example. Consider for $s > 1$ the function $\Psi : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$\Psi(\zeta) = \prod_{j=1}^{\infty} \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta}.$$

We claim that Ψ is the Fourier-Laplace transform of some function in $\mathcal{G}_c^s$, and we shall prove this by using the main result of this section.

We begin by showing that Ψ extends to an entire function. In view of Proposition 2.21, it suffices to show that the sum

$$\sum_{j=1}^{\infty} \left| 1 - \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \quad (3.19)$$

converges uniformly on compact subsets of $\mathbb{C}$. Firstly note that, given that $|\zeta| \leq 1$, the Taylor series expansion of $\frac{\sin(\zeta)}{\zeta}$ gives

$$\left| 1 - \frac{\sin(\zeta)}{\zeta} \right| = \left| \sum_{k=1}^{\infty} \frac{(-1)^k \zeta^{2k}}{(2k+1)!} \right| \leq |\zeta| \sum_{k=1}^{\infty} \frac{|\zeta|^{2k-1}}{(2k+1)!} \leq |\zeta| \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \lesssim |\zeta|,$$

where the final inequality follows since the sum clearly converges. Thus, if we replace ζ by $j^{-s}\zeta$, we obtain

$$\left| 1 - \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \lesssim j^{-s} |\zeta|,$$

given that $|\zeta| \leq j^s$, for all $j \geq 1$.

Now let $K \subset \mathbb{C}$ be compact. Then K must be bounded, so that all $\zeta \in K$ satisfy $|\zeta| \leq C_K$ for some constant $C_K > 0$. Furthermore, the same reasoning shows that we can choose N so large that whenever $j \geq N$, every $\zeta \in K$ satisfies $|\zeta| \leq j^s$. Then

$$\begin{aligned} \sum_{j=1}^{\infty} \left| 1 - \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| &= \sum_{j=1}^{N-1} \left| 1 - \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| + \sum_{j=N}^{\infty} \left| 1 - \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \\ &\lesssim C_N + \sum_{j=N}^{\infty} j^{-s} |\zeta| \leq C_N + C_K \sum_{j=N}^{\infty} j^{-s} < \infty, \end{aligned}$$

where the final step follows since C_N is just a fixed constant, and the infinite sum converges since $s > 1$. It follows that the sum (3.19) converges uniformly on compact subsets of $\mathbb{C}$, giving that Ψ is entire.

Next, we must show that Ψ satisfies the estimate (3.3). To do this, we will mimick the argument used in the proof of the Paley-Wiener Theorem. We have

$$\left| \frac{\sin(\zeta)}{\zeta} \right| = \left| \frac{e^{i\zeta} - e^{-i\zeta}}{2i\zeta} \right| \leq \frac{|e^{-\eta} e^{i\xi}| + |e^{\eta} e^{-i\xi}|}{2|\zeta|} = \frac{|e^{-\eta}| + |e^{\eta}|}{2|\zeta|} \leq \frac{e^{|\eta|}}{|\zeta|},$$

for all $\zeta \in \mathbb{C} \setminus \{0\}$. Hence, replacing ζ by $j^{-s}\zeta$ gives

$$\left| \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \leq \frac{\exp(j^{-s}|\eta|)}{j^{-s}|\zeta|}, \quad \zeta \in \mathbb{C} \setminus \{0\}, \quad (3.20)$$

for all $j \geq 1$. Furthermore, we can get another estimate by noting that

$$\frac{\sin(\zeta)}{\zeta} = \frac{e^{i\zeta} - e^{-i\zeta}}{2i\zeta} = \frac{1}{2} \int_{-1}^1 e^{i\zeta t} dt,$$

so that

$$\left| \frac{\sin(\zeta)}{\zeta} \right| = \left| \frac{1}{2} \int_{-1}^1 e^{i\zeta t} dt \right| \leq \frac{1}{2} \int_{-1}^1 |e^{-\eta t}| dt \leq \frac{1}{2} \int_{-1}^1 e^{|\eta|} dt = e^{|\eta|},$$

for all $\zeta \in \mathbb{C}$. Thus, for all $j \geq 1$ we also have

$$\left| \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \leq \exp(j^{-s}|\eta|), \quad \zeta \in \mathbb{C}. \quad (3.21)$$

Now we split the product into two parts, and apply the estimates (3.20) and (3.21) respectively, so that for any $n \geq 0$ and $\zeta \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} |\Psi(\zeta)| &= \prod_{j=1}^n \left| \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \prod_{j=n+1}^{\infty} \left| \frac{\sin(j^{-s}\zeta)}{j^{-s}\zeta} \right| \leq \prod_{j=1}^n \frac{\exp(j^{-s}|\eta|)}{j^{-s}|\zeta|} \prod_{j=n+1}^{\infty} \exp(j^{-s}|\eta|) \\ &= \frac{n!^s}{|\zeta|^n} \prod_{j=1}^{\infty} \exp(j^{-s}|\eta|) \leq \frac{n^{sn}}{|\zeta|^n} \exp\left(\sum_{j=1}^{\infty} j^{-s}|\eta|\right) = \frac{n^{sn}}{|\zeta|^n} e^{R_s|\eta|}, \end{aligned}$$

where $R_s = \sum_{j=1}^{\infty} j^{-s}$ converges since $s > 1$. Hence,

$$|\zeta|^n |\Psi(\zeta)| \leq n^{sn} e^{R_s|\eta|},$$

and it follows from Lemma 3.9 that we can find a constant $M > 0$ such that

$$\sup_{k \in \mathbb{N}} \left\{ \frac{\lambda^k}{k!} |\zeta|^{\frac{k}{s}} |\Psi(\zeta)| \right\} \lesssim \sup_{n \in \mathbb{N}} \left\{ \left(\frac{M\lambda}{n} \right)^{sn} |\zeta|^n |\Psi(\zeta)| \right\} \leq \sup_{n \in \mathbb{N}} (M\lambda)^{sn} e^{R_s|\eta|}.$$

It then follows in the same way as in the proof of Theorem 3.8 that

$$|\Psi(\zeta)| \lesssim \exp\left(-\lambda |\zeta|^{\frac{1}{s}} + R_s|\eta|\right),$$

for every $\zeta \in \mathbb{C} \setminus \{0\}$ and some positive constant λ . In fact, this estimate trivially holds even at $\zeta = 0$.

Hence, Ψ satisfies the requirements of the Paley-Wiener Theorem for $\mathcal{G}_c^s$, and is therefore the Fourier-Laplace transform of a compactly supported Gevrey function. In particular, this shows that the Fourier inverse of Ψ , i.e.

$$f(x) = \int_{\mathbb{R}} e^{ix\xi} \prod_{j=1}^{\infty} \frac{\sin(j^{-s}\xi)}{j^{-s}\xi} d\mu_1(\xi),$$

belongs to $\mathcal{G}_c^s$ and has its support contained in $[-R_s, R_s]$. $\triangle$

4. Gevrey Ultradistributions

4.1. The spaces $\mathcal{D}'_s$ and $\mathcal{E}'_s$

Following the recipe from classical distribution theory, we can also consider the Gevrey analogues of Schwartz distributions, i.e. linear continuous functionals on the spaces $\mathcal{G}^s$ and $\mathcal{G}^s_c$ for $s > 1$. To do this, we should define what we mean by convergence in $\mathcal{G}^s$ and $\mathcal{G}^s_c$. Definition 3.1 naturally induces the following mode of convergence.

4.1 Definition. Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $\{\phi_j\}_{j=1}^\infty$ be a sequence in $\mathcal{G}^s(\Omega)$ and let $\phi \in \mathcal{G}^s(\Omega)$. Then we say that $\phi_j \rightarrow \phi$ in $\mathcal{G}^s(\Omega)$ as $j \rightarrow \infty$ whenever for every compact set $K \subseteq \Omega$ we can find a constant $\tau = \tau_K > 0$ such that

$$\sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha \phi_j(x) - \partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s} \rightarrow 0, \quad \text{as } j \rightarrow \infty. \quad (4.1)$$

We say that $\phi_j \rightarrow \phi$ in $\mathcal{G}^s_c(\Omega)$ as $j \rightarrow \infty$ whenever there exists a compact set $K \subseteq \Omega$ containing the supports of ϕ and all ϕ_j , and a constant $\tau > 0$ such that (4.1) holds.

4.2 Definition ($\mathcal{D}'_s$ and $\mathcal{E}'_s$). Let $\Omega \subseteq \mathbb{R}^d$ be open and let $s > 1$.

- (i) The space $\mathcal{D}'_s(\Omega)$ is the set of all linear forms $u : \mathcal{G}^s_c(\Omega) \rightarrow \mathbb{C}$ that are continuous in the sense that $u(\phi_j) \rightarrow u(\phi)$ whenever $\phi_j \rightarrow \phi$ in $\mathcal{G}^s_c(\Omega)$ as $j \rightarrow \infty$.
- (ii) The space $\mathcal{E}'_s(\Omega)$ is the set of all linear forms $u : \mathcal{G}^s(\Omega) \rightarrow \mathbb{C}$ that are continuous in the sense that $u(\phi_j) \rightarrow u(\phi)$ whenever $\phi_j \rightarrow \phi$ in $\mathcal{G}^s(\Omega)$ as $j \rightarrow \infty$.

The elements of $\mathcal{D}'_s$ are called *ultradistributions*. We will discuss $\mathcal{E}'_s$ later. As in the classical case, the continuity requirement for $\mathcal{D}'_s$ may be replaced by another condition as shown by the following proposition.

4.3 Proposition. A linear form $u : \mathcal{G}^s_c(\Omega) \rightarrow \mathbb{C}$ is an element of $\mathcal{D}'_s(\Omega)$ if and only if for every compact set $K \subseteq \Omega$ and every $\tau > 0$, there exists a constant $C_\tau > 0$ such that

$$|u(\phi)| \leq C_\tau \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s},$$

for all $\phi \in \mathcal{G}^s_c(\Omega)$ with $\text{supp}(\phi) \subseteq K$.

Proof. See [Rod93, Proposition 1.5.4].

Q.E.D.

Let us hereafter assume that $\Omega = \mathbb{R}^d$. The name ultradistribution is motivated from the fact that every Schwartz distribution also belongs to $\mathcal{D}'_s$. In fact, if $u \in \mathcal{D}'$ and $\phi \in \mathcal{G}_c^s \subseteq \mathcal{C}_c^\infty$ with $\text{supp}(\phi) \subseteq K$, we obtain for arbitrary $\tau > 0$ the estimate

$$\begin{aligned} |u(\phi)| &\lesssim \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \phi(x)| = \sum_{|\alpha| \leq N} \sup_{x \in K} \frac{|\partial^\alpha \phi(x)|}{(2\tau)^{|\alpha|} \alpha!^s} (2\tau)^{|\alpha|} \alpha!^s \\ &\leq (2\tau)^N N!^s \sup_{\substack{|\alpha| \leq N \\ x \in K}} \frac{|\partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s} \sum_{|\alpha| \leq N} 2^{-|\alpha|} \leq C_\tau \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s}, \end{aligned}$$

so that u also defines an element in $\mathcal{D}'_s$.

4.4 Example. We identify a function $f \in \mathcal{L}_{\text{loc}}^1$ with the functional $u_f : \mathcal{G}_c^s \rightarrow \mathbb{C}$ defined by

$$u_f(\phi) = \int_{\mathbb{R}^d} f(x) \phi(x) d\mu_d(x),$$

which belongs to $\mathcal{D}'$ and hence to $\mathcal{D}'_s$ as well. $\triangle$

However, one can find ultradistributions which do not belong to $\mathcal{D}'$, as the following example suggests.

4.5 Example. Let $u : \mathcal{G}_c^s \rightarrow \mathbb{C}$ be the functional defined by

$$u = \sum_{\alpha \in \mathbb{N}^d} \frac{\partial^\alpha \delta}{\alpha!^{s+1}},$$

where δ is the Dirac delta. More formally, if $\phi \in \mathcal{G}_c^s$ then $u(\phi)$ is the number

$$u(\phi) = \sum_{\alpha \in \mathbb{N}^d} (-1)^{|\alpha|} \frac{\partial^\alpha \phi(0)}{\alpha!^{s+1}}.$$

This construction is well-defined, since the fact that $\phi \in \mathcal{G}_c^s \subseteq \mathcal{G}^s$ allows us to find a constant $\tau_0 > 0$ such that $|\partial^\alpha \phi(0)| \lesssim \tau_0^{|\alpha|} \alpha!^s$, and then

$$|u(\phi)| \leq \sum_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(0)|}{\alpha!^{s+1}} \lesssim \sum_{\alpha \in \mathbb{N}^d} \frac{\tau_0^{|\alpha|}}{\alpha!} \leq \sum_{\alpha \in \mathbb{N}^d} \frac{(2d\tau_0)^{|\alpha|}}{2^{|\alpha|} |\alpha|!} \leq e^{2d\tau_0} \sum_{\alpha \in \mathbb{N}^d} 2^{-|\alpha|} < \infty,$$

where we use the inequalities $|\alpha|! \leq d^{|\alpha|} \alpha!$ and $\frac{t^n}{n!} \leq e^t$ with $t = 2d\tau_0$ and $n = |\alpha|$.

The linearity of u follows since u is a linear combination of derivatives Dirac deltas which are also linear. To show continuity, let K be compact, let $\phi \in \mathcal{G}_c^s$ with $\text{supp}(\phi) \subseteq K$, and let $\tau > 0$ be arbitrary. Then we have

$$\frac{1}{\alpha!} \leq \frac{d^{|\alpha|}}{|\alpha|!} \leq e^{2d\tau} \frac{d^{|\alpha|}}{(2d\tau)^{|\alpha|}} = \frac{C_\tau}{(2\tau)^{|\alpha|}},$$

with $C_\tau = e^{2d\tau}$. Thus,

$$\begin{aligned} |u(\phi)| &\leq \sum_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(0)|}{\alpha!^{s+1}} \leq C_\tau \sum_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(0)|}{(2\tau)^{|\alpha|} \alpha!^s} \leq C_\tau \sup_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(0)|}{\tau^{|\alpha|} \alpha!^s} \sum_{\alpha \in \mathbb{N}^d} 2^{-|\alpha|} \\ &\lesssim C_\tau \sup_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(0)|}{\tau^{|\alpha|} \alpha!^s} \leq C_\tau \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s}, \end{aligned}$$

for every $\tau > 0$ and $\phi \in \mathcal{G}_c^s$ with $\text{supp}(\phi) \subseteq K$. Note that the final inequality follows trivially if K contains the origin, but it holds even if K does not contain the origin since in that case $\partial^\alpha \phi(0) = 0$ for all $\alpha \in \mathbb{N}^d$. Hence, Proposition 4.3 shows that $u \in \mathcal{D}'_s$.

However, $u \notin \mathcal{D}'$ as u is not even well-defined for all $\phi \in \mathcal{C}_c^\infty$. In fact, Borel's lemma (see e.g. [Hör90, Theorem 1.2.6]) guarantees the existence of a $\phi \in \mathcal{C}_c^\infty$ such that $\partial^\alpha \phi(0) = \alpha!^{s+1}$, and then the sum in $u(\phi)$ clearly diverges. $\triangle$

We will mostly be interested in ultradistributions whose support is compact.

4.6 Definition. Let $\omega \subseteq \mathbb{R}^d$ be open. Then an ultradistribution $u \in \mathcal{D}'_s$ is said to vanish on ω if for every $\phi \in \mathcal{G}_c^s$ with $\text{supp}(\phi) \subseteq \omega$ we have $u(\phi) = 0$. The support of an ultradistribution $u \in \mathcal{D}'_s$ is defined as the complement of the largest open set ω on which u vanishes.

4.7 Example. Let u be the ultradistribution from Example 4.5. Since u is a series of derivatives of Dirac deltas, it follows that the support of u is the singleton $\{0\}$, so that u is a compactly supported ultradistribution. $\triangle$

As in the case of Schwartz distributions, it can in fact be shown that every ultradistribution whose support is compact extends uniquely to a linear functional on $\mathcal{G}^s$. Indeed, if $u \in \mathcal{D}'_s$ with $\text{supp}(u) \subseteq K$, then we can find a cutoff function $\chi \in \mathcal{G}_c^s$ such that $\chi = 1$ on a neighbourhood of K , and then

$$u(\phi) = u(\chi\phi), \tag{4.2}$$

since u does not care what happens outside a neighbourhood of its support. The right-hand side of (4.2) is clearly well-defined for every $\phi \in \mathcal{G}^s$ in view of Proposition 3.3, and it is not hard to see that this definition is independent of the chosen cutoff function. Conversely, the restriction of $u \in \mathcal{E}'_s$ to $\mathcal{G}_c^s$ defines an ultradistribution since $\mathcal{G}_c^s$ convergence implies convergence in $\mathcal{G}^s$ as well. Moreover, the continuity of u forces this restriction to have compact support. This motivates the following result.

4.8 Proposition. *The space $\mathcal{E}'_s$ is identical to the space of elements in $\mathcal{D}'_s$ whose support is compact.*

Proof. For all the details of a more general version of this proposition, see [Kom73, Theorem 5.9]. Q.E.D.

The identification is of course understood in the sense of the formula (4.2). In view of this, we will usually refer to elements of $\mathcal{E}'_s$ as compactly supported ultradistributions, and the support of $u \in \mathcal{E}'_s$ will be understood in the sense of Definition 4.6. We also obtain the following analogue of Proposition 4.3.

4.9 Proposition. *A linear form $u : \mathcal{G}^s \rightarrow \mathbb{C}$ is an element of $\mathcal{E}'_s$ with support in a compact set $K \subseteq \mathbb{R}^d$ if and only if for every cutoff function $\chi \in \mathcal{G}^s_c$ such that $\chi = 1$ on a neighbourhood of K , and every $\tau > 0$, there exists a constant $C_\tau > 0$ such that*

$$|u(\phi)| \leq C_\tau \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in \mathbb{R}^d}} \frac{|\partial^\alpha (\phi(x)\chi(x))|}{\tau^{|\alpha|} \alpha!^s},$$

for all $\phi \in \mathcal{G}^s$.

Let us note that the support of the cutoff functions in the preceding proposition is automatically slightly larger than the support of u itself.

We continue by defining the operations on ultradistributions. Most of the following definitions are direct analogues of the classical definitions.

4.10 Definition. Let $u \in \mathcal{D}'_s$ and $\phi \in \mathcal{G}^s_c$, or $u \in \mathcal{E}'_s$ and $\phi \in \mathcal{G}^s$. Let also $\psi \in \mathcal{G}^s$. Then

$$\psi u(\phi) := u(\psi\phi), \quad \partial^\alpha u(\phi) := (-1)^{|\alpha|} u(\partial^\alpha \phi), \quad \text{and} \quad (u * \phi)(x) := u(\phi(x - \cdot)).$$

The right-hand side in all three cases are well defined in view of Proposition 3.3. In view of our identification of locally integrable functions with distributions from Example 4.4, and since the function $x \mapsto e^{-i\langle x, \zeta \rangle}$ belongs to $\mathcal{A} \subseteq \mathcal{G}^s$ for every fixed $\zeta \in \mathbb{C}^d$, the following definition also seems natural.

4.11 Definition. The Fourier-Laplace transform of a compactly supported ultradistribution $u \in \mathcal{E}'_s$ is the function $\Psi : \mathbb{C}^d \rightarrow \mathbb{C}$ defined as

$$\Psi(\zeta) := u\left(e^{-i\langle \cdot, \zeta \rangle}\right).$$

The Fourier transform $\widehat{u}$ of u is defined as the restriction of Ψ to $\mathbb{R}^d$.

4.12 Example. The Fourier-Laplace transform of the ultradistribution from Example 4.5 is given by

$$\Psi(\zeta) = \sum_{\alpha \in \mathbb{N}^d} \frac{i^{|\alpha|} \zeta^\alpha}{\alpha!^{s+1}}.$$

Notice that Ψ is an entire function in view of Proposition 2.14, as it is defined by a power series which converges absolutely in a neighbourhood of every $\zeta \in \mathbb{C}^d$. $\triangle$

The definitions of the convolution and the Fourier-Laplace transform suggest that we might find interest in knowing how ultradistributions behave when acting on functions depending on some other parameters. The following proposition gives us the answer. The proof of the proposition seems to be hard to find in the literature, and it differs from its classical counterpart, so let us also state it since we will apply the proposition on several occasions later.

4.13 Proposition. *Let $u \in \mathcal{D}'_s(\mathbb{R}^{d_1})$ and let $\phi \in \mathcal{G}^s(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be such that $x \mapsto \phi(x, y)$ is supported in a compact set $K \subseteq \mathbb{R}^{d_1}$ for every $y \in \mathbb{R}^{d_2}$. Then the function*

$$\Phi(y) = u(\phi(\cdot, y))$$

is infinitely many times differentiable on $\mathbb{R}^{d_2}$ and its partial derivatives are given by

$$\partial_y^\alpha \Phi(y) = \partial_y^\alpha u(\phi(\cdot, y)) = u(\partial_y^\alpha \phi(\cdot, y)),$$

for every multi-index $\alpha \in \mathbb{N}^{d_2}$. Moreover, $\Phi \in \mathcal{G}^s(\mathbb{R}^{d_2})$.

Remark. If $u \in \mathcal{E}'_s(\mathbb{R}^{d_1})$, then one can reach the same conclusions by only assuming that $\phi \in \mathcal{G}^s(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$.

Proof. We first note that Φ is well-defined for every $y \in \mathbb{R}^{d_2}$ due to the assumptions. Since $y \mapsto \phi(x, y)$ belongs to $\mathcal{G}^s \subseteq \mathcal{C}^\infty$ for every fixed $x \in \mathbb{R}^{d_1}$, Taylor's formula gives

$$\phi(x, y + h) = \phi(x, y) + \sum_{j=1}^{d_2} h_j \frac{\partial \phi}{\partial y_j}(x, y) + R(x, y, h),$$

for $h \in \mathbb{R}^{d_2}$, and where R can be represented as

$$R(x, y, h) = 2 \int_0^1 (1 - \theta) \sum_{|\beta|=2} \partial_y^\beta \phi(x, y + \theta h) \frac{h^\beta}{\beta!} d\theta.$$

Then by the linearity of u ,

$$u(\phi(\cdot, y + h)) = u(\phi(\cdot, y)) + \sum_{j=1}^{d_2} h_j u(\partial_{y_j} \phi(\cdot, y)) + u(R(\cdot, y, h)). \quad (4.3)$$

To show that Φ is differentiable, it therefore suffices to show that

$$u\left(\frac{R(\cdot, y, h)}{|h|}\right) \rightarrow 0, \quad \text{whenever } h \rightarrow 0.$$

Proposition 4.3 gives the estimate

$$\left| u\left(\frac{R(\cdot, y, h)}{|h|}\right) \right| \leq C_\tau \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial_x^\alpha R(x, y, h)|}{\tau^{|\alpha|} \alpha! |h|}, \quad (4.4)$$

for every $\tau > 0$ and compact set $K \subseteq \mathbb{R}^{d_1}$. We have

$$\begin{aligned} |\partial_x^\alpha R(x, y, h)| &= \left| 2 \int_0^1 (1 - \theta) \sum_{|\beta|=2} \partial_x^\alpha \partial_y^\beta \phi(x, y + \theta h) \frac{h^\beta}{\beta!} d\theta \right| \\ &\lesssim |h|^2 \sup_{\substack{|\beta|=2 \\ \theta \in [0,1]}} |\partial_x^\alpha \partial_y^\beta \phi(x, y + \theta h)|, \end{aligned}$$

where we are allowed to differentiate under the integral since the integration is taken over a compact set and the integrand is well-behaved. If $|h| \leq 1$, the supremum of $y + \theta h$ over $\theta \in [0, 1]$ can be replaced by a supremum of t on the ball $\bar{B}_1(y)$, so that

$$\sup_{x \in K} |\partial_x^\alpha R(x, y, h)| \lesssim |h|^2 \sup_{|\beta|=2} \sup_{\substack{x \in K \\ t \in \bar{B}_1(y)}} |\partial_x^\alpha \partial_t^\beta \phi(x, t)|.$$

Since $\phi \in \mathcal{G}^s(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, we can find a constant $M > 0$ such that

$$\sup_{\substack{x \in K \\ t \in \bar{B}_1(y)}} |\partial_x^\alpha \partial_t^\beta \phi(x, t)| \lesssim M^{|\alpha|+|\beta|} (\alpha + \beta)!^s \leq (2^s M)^{|\alpha|+|\beta|} \alpha!^s \beta!^s,$$

where the final inequality follows since $(\alpha + \beta)! \leq 2^{|\alpha|+|\beta|} \alpha! \beta!$. Thus,

$$\sup_{x \in K} |\partial_x^\alpha R(x, y, h)| \lesssim |h|^2 \sup_{|\beta|=2} \left\{ (2^s M)^{|\alpha|+|\beta|} \alpha!^s \beta!^s \right\} \lesssim |h|^2 (2^s M)^{|\alpha|} \alpha!^s,$$

and (4.4) therefore implies

$$\begin{aligned} \left| u \left(\frac{R(\cdot, y, h)}{|h|} \right) \right| &\lesssim C_\tau \sup_{\alpha \in \mathbb{N}^d} \frac{|h|^2 (2^s M)^{|\alpha|} \alpha!^s}{\tau^{|\alpha|} \alpha!^s |h|} \\ &\leq C_\tau |h| \sup_{\alpha \in \mathbb{N}^d} \left\{ \left(\frac{2^s M}{\tau} \right)^{|\alpha|} \right\}. \end{aligned}$$

In particular, if we choose $\tau = 2^s M$, the expression inside the supremum becomes equal to 1, so that

$$\left| u \left(\frac{R(\cdot, y, h)}{|h|} \right) \right| \lesssim |h| \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

Hence Φ is differentiable on $\mathbb{R}^{d_2}$. The formula for the partial derivatives of first order now follows immediately from (4.3).

That Φ is infinitely many times differentiable on $\mathbb{R}^{d_2}$ and that the formula for the general partial derivatives holds is obtained inductively by iterating the proof above.

Finally, for every compact set $K_0 \subseteq \mathbb{R}^{d_2}$, we can find a constant $L > 0$ such that

$$\sup_{\substack{x \in K \\ y \in K_0}} |\partial_x^\alpha \partial_y^\beta \phi(x, y)| \lesssim L^{|\alpha|+|\beta|} (\alpha + \beta)!^s,$$

since $\phi \in \mathcal{G}^s(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Then we obtain from the formula for the partial derivatives that

$$\begin{aligned} \sup_{y \in K_0} |\partial_y^\beta \Phi(y)| &= \sup_{y \in K_0} |u(\partial_y^\beta \phi(\cdot, y))| \leq C_\tau \sup_{\alpha \in \mathbb{N}^d} \sup_{\substack{x \in K \\ y \in K_0}} \frac{|\partial_x^\alpha \partial_y^\beta \phi(x, y)|}{\tau^{|\alpha|} \alpha!^s} \\ &\lesssim C_\tau \sup_{\alpha \in \mathbb{N}^d} \frac{L^{|\alpha|+|\beta|} (\alpha + \beta)!^s}{\tau^{|\alpha|} \alpha!^s} \leq C_\tau (2^s L)^{|\beta|} \beta!^s \sup_{\alpha \in \mathbb{N}^d} \left(\frac{2^s L}{\tau} \right)^{|\alpha|}. \end{aligned}$$

If we choose $\tau = 2^s L$, the supremum over α becomes equal to 1, and we see that $\Phi \in \mathcal{G}^s(\mathbb{R}^{d_2})$. Q.E.D.

Using this proposition, we can now show that the convolution operation behaves quite nicely for ultradistributions.

4.14 Proposition. *If $\phi \in \mathcal{G}_c^s$ and $u \in \mathcal{D}'_s$, or else if $\phi \in \mathcal{G}^s$ and $u \in \mathcal{E}'_s$, then $u * \phi \in \mathcal{G}^s$. If the support of both ϕ and u is compact, then the support of $u * \phi$ is compact as well.*

Proof. The first part of the proposition is an immediate consequence of Proposition 4.13. The compactness of $\text{supp}(u * \phi)$ when both ϕ and u are compactly supported follows as in the classical case from the formula $\text{supp}(u * \phi) \subseteq \text{supp}(u) + \text{supp}(\phi)$, see e.g. [Hör90, Theorem 4.1.1]. Q.E.D.

The analogues of certain very useful regularisation results from classical distribution theory hold for ultradistributions as well. We will need the following definition to state these results.

4.15 Definition. Let $\{u_j\}_{j=1}^\infty \subseteq \mathcal{E}'_s$ and $u \in \mathcal{E}'_s$. Then we say that $u_j \rightarrow u$ in $\mathcal{E}'_s$ as $j \rightarrow \infty$ whenever $u_j(\phi) \rightarrow u(\phi)$ as $j \rightarrow \infty$ for every $\phi \in \mathcal{G}^s$.

4.16 Proposition. *Let $\phi \in \mathcal{G}^s$, $u \in \mathcal{E}'_s$, and let $\{\psi_\epsilon\}_{\epsilon>0}$ be an approximate identity of Gevrey class. Then*

- (a) $\phi * \psi_\epsilon \rightarrow \phi$ in $\mathcal{G}^s$ as $\epsilon \rightarrow 0^+$;
- (b) $u * \psi_\epsilon \rightarrow u$ in $\mathcal{E}'_s$ as $\epsilon \rightarrow 0^+$.

Proof. (a) We need to show that

$$\sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha (\phi * \psi_\epsilon)(x) - \partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+,$$

for every compact set $K \subseteq \mathbb{R}^d$ and some constant $\tau > 0$. Utilizing Proposition 4.13, we find that

$$\partial^\alpha (\phi * \psi_\epsilon) = (\partial^\alpha \phi) * \psi_\epsilon,$$

and since $\int \psi_\epsilon d\mu_d = 1$, we get

$$\begin{aligned}
|\partial^\alpha(\phi * \psi_\epsilon)(x) - \partial^\alpha \phi(x)| &= \left| \int_{\mathbb{R}^d} \partial^\alpha \phi(x-y) \psi_\epsilon(y) d\mu_d(y) - \partial^\alpha \phi(x) \int_{\mathbb{R}^d} \psi_\epsilon(y) d\mu_d(y) \right| \\
&\leq \int_{\bar{B}_\epsilon} |\partial^\alpha \phi(x-y) - \partial^\alpha \phi(x)| |\psi_\epsilon(y)| d\mu_d(y) \\
&\leq \sup_{y \in \bar{B}_\epsilon} |\partial^\alpha \phi(x-y) - \partial^\alpha \phi(x)|.
\end{aligned} \tag{4.5}$$

Since $\phi \in \mathcal{C}^\infty$, we also have $\partial^\alpha \phi \in \mathcal{C}^\infty$, so that Taylor's formula gives the estimate

$$\begin{aligned}
|\partial^\alpha \phi(x-y) - \partial^\alpha \phi(x)| &= \left| \int_0^1 \sum_{|\beta|=1} \frac{(-y)^\beta}{\beta!} \partial^{\alpha+\beta} \phi(x-\theta y) d\theta \right| \\
&\leq |y| \int_0^1 \sum_{|\beta|=1} |\partial^{\alpha+\beta} \phi(x-\theta y)| d\theta,
\end{aligned}$$

and thus (4.5) becomes

$$|\partial^\alpha(\phi * \psi_\epsilon)(x) - \partial^\alpha \phi(x)| \leq \epsilon \sum_{|\beta|=1} \sup_{\substack{y \in \bar{B}_\epsilon \\ \theta \in [0,1]}} |\partial^{\alpha+\beta} \phi(x-\theta y)|.$$

Given that $\epsilon \leq 1$, the supremum over y and θ is bounded by the supremum over $\bar{B}_1$. Since $\phi \in \mathcal{G}^s$ by assumption, we may therefore for every compact set $K \subseteq \mathbb{R}^d$ find a constant $M > 0$ such that

$$\sum_{|\beta|=1} \sup_{\substack{x \in K \\ y \in \bar{B}_1}} |\partial^{\alpha+\beta} \phi(x-\theta y)| \lesssim \sum_{|\beta|=1} M^{|\alpha|+|\beta|} (\alpha+\beta)!^s \lesssim (2^s M)^{|\alpha|} \alpha!^s,$$

where the final inequality follows again since $(\alpha+\beta)! \leq 2^{|\alpha|+|\beta|} \alpha! \beta!$. Thus,

$$\sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha(\phi * \psi_\epsilon)(x) - \partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s} \lesssim \epsilon \sup_{\alpha \in \mathbb{N}^d} \frac{(2^s M)^{|\alpha|} \alpha!^s}{\tau^{|\alpha|} \alpha!^s} = \epsilon \sup_{\alpha \in \mathbb{N}^d} \left(\frac{2^s M}{\tau} \right)^{|\alpha|}.$$

Letting $\tau = 2^s M$, this becomes

$$\sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K}} \frac{|\partial^\alpha(\phi * \psi_\epsilon)(x) - \partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s} \lesssim \epsilon \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+.$$

(b) We follow the proof of [Hör90, Theorem 4.1.4]. Define temporarily $\tilde{\phi}(x) = \phi(-x)$, and note that for every $v \in \mathcal{E}'_s$ and $\phi \in \mathcal{G}^s$ we can write $v(\phi) = (v * \tilde{\phi})(0)$. Then

$$(u * \psi_\epsilon)(\phi) = \left((u * \psi_\epsilon) * \tilde{\phi} \right)(0) = \left(u * (\psi_\epsilon * \tilde{\phi}) \right)(0) = u \left(\tilde{\psi}_\epsilon * \phi \right) \rightarrow u(\phi),$$

as $\epsilon \rightarrow 0^+$, by part (a) and the continuity of u .

Q.E.D.

Remark. In the second equality in the proof of part (b), we use the fact that the convolution is associative. The proof that this is true requires a theorem of Fubini type for ultradistributions. However, for the purposes of this work we simply accept that this associativity property holds.

4.2. The Paley-Wiener Theorem for $\mathcal{E}'_s$

We are now ready to discuss our main result. In the classical case, the space $\mathcal{C}_c^\infty$ is characterized by (among other things) the polynomial decay of the Fourier-Laplace transforms of its elements, whereas polynomial decay is replaced by polynomial growth in the case of $\mathcal{E}'$. Since we have shown that $\mathcal{G}_c^s$ is characterized by the exponential decay of the Fourier-Laplace transform, one may expect exponential growth in the case of $\mathcal{E}'_s$. That this is true is the content of the following Paley-Wiener Theorem.

4.17 Paley-Wiener Theorem for $\mathcal{E}'_s$. *Let $K \subseteq \mathbb{R}^d$ be a compact convex set. An ultradistribution u belongs to $\mathcal{E}'_s$ and has its support contained in K if and only if its Fourier-Laplace transform Ψ is an entire function that satisfies for every $\lambda > 0$ and some constant $C_\lambda > 0$ the estimate*

$$|\Psi(\zeta)| \leq C_\lambda \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_K(\operatorname{Im} \zeta) \right), \quad (4.6)$$

for every $\zeta \in \mathbb{C}^d$.

Before continuing, let us state a result whose importance will be clarified after we prove it.

4.18 Lemma. *Define $\Pi : \mathbb{C} \rightarrow \mathbb{C}$ for $\lambda > 0$ by*

$$\Pi(\zeta) = \prod_{j=1}^{\infty} \left(1 - \frac{i\lambda^s \zeta}{j^s} \right).$$

Then

- (a) Π is entire;
- (b) $\frac{1}{\Pi} \in \mathcal{H}(\mathbb{C}_+) \cap \mathcal{C}(\overline{\mathbb{C}}_+)$;
- (c) Π satisfies for all $\zeta \in \overline{\mathbb{C}}_+$, and some $L > 0$ the estimates

$$\exp \left(\lambda |\zeta|^{\frac{1}{s}} \right) \lesssim |\Pi(\zeta)| \lesssim \exp \left(L\lambda |\zeta|^{\frac{1}{s}} \right).$$

Proof. (a) In view of Proposition 2.21, it suffices to show that

$$\sum_{j=1}^{\infty} \frac{\lambda^s |\zeta|}{j^s} < \infty$$

uniformly on compact sets. But this follows immediately since

$$\sum_{j=1}^{\infty} \frac{\lambda^s |\zeta|}{j^s} = \lambda^s |\zeta| \sum_{j=1}^{\infty} \frac{1}{j^s} \lesssim |\zeta| < \infty,$$

where the final inequality follows since $|\zeta|$ is bounded whenever ζ belongs to a compact subset of $\mathbb{C}^d$, and thus Π is entire.

(b) Notice that all zeros of Π lie on the lower half-plane, namely on the negative imaginary axis. It follows that $\frac{1}{\Pi}$ has no singularities on $\overline{\mathbb{C}}_+$ and is therefore holomorphic on $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}}_+$ by part (a).

(c) For the first inequality, we note that since $\zeta \in \overline{\mathbb{C}}_+$, the modulus of every factor in the product is greater than or equal to 1, so that

$$\begin{aligned} |\Pi(\zeta)| &= \sup_{n \geq 0} \prod_{j=1}^n \left| 1 - \frac{i\lambda^s \zeta}{j^s} \right| \geq \sup_{n \geq 0} \prod_{j=1}^n \frac{\lambda^s |\zeta|}{j^s} = \sup_{n \geq 0} \left\{ \frac{\lambda^{sn} |\zeta|^n}{n!^s} \right\} \geq \left(\sup_{n \geq 0} \left\{ \frac{\lambda^n |\zeta|^{\frac{n}{s}}}{n!} \right\} \right)^s \\ &\gtrsim \left(\sup_{n \geq 0} \left\{ \frac{\lambda^n |\zeta|^{\frac{n}{s}}}{n!} \right\} \sum_{n=0}^{\infty} s^{-n} \right)^s \geq \left(\sum_{n=0}^{\infty} \frac{\left(s^{-1} \lambda |\zeta|^{\frac{1}{s}} \right)^n}{n!} \right)^s = \exp \left(\lambda |\zeta|^{\frac{1}{s}} \right). \end{aligned}$$

For the second inequality, we note that

$$\begin{aligned} \log |\Pi(\zeta)| &= \log \left(\prod_{j=1}^{\infty} \left| 1 - \frac{i\lambda^s \zeta}{j^s} \right| \right) \leq \log \left(\prod_{j=1}^{\infty} \left(1 + \frac{\lambda^s |\zeta|}{j^s} \right) \right) = \sum_{j=1}^{\infty} \log \left(1 + \frac{\lambda^s |\zeta|}{j^s} \right) \\ &\leq \int_0^{\infty} \log \left(1 + \frac{\lambda^s |\zeta|}{x^s} \right) dx = \lambda |\zeta|^{\frac{1}{s}} \int_0^{\infty} \log \left(1 + \frac{1}{x^s} \right) dx, \end{aligned}$$

where the sum is bounded by the integral since the logarithm inside the sum is non-negative and decreasing with respect to j , and the final equality follows by the variable change $x \mapsto x \left(\lambda |\zeta|^{\frac{1}{s}} \right)^{-1}$ in the integral. Since the final integral converges (see Appendix A), we obtain

$$\log |\Pi(\zeta)| \leq L \lambda |\zeta|^{\frac{1}{s}},$$

for every $\zeta \in \overline{\mathbb{C}}_+$ (in fact, every $\zeta \in \mathbb{C}$). Exponentiating both sides gives the result. *Q.E.D.*

A remark is in order before we begin with the proof of Theorem 4.17. In the proof to come, we will apply Proposition 4.9 to the Fourier-Laplace transform Ψ to obtain the estimate

$$|\Psi(\zeta)| \leq C_{\lambda, \epsilon} \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_K(\eta) + \epsilon |\eta| \right), \quad (4.7)$$

for every $\lambda > 0$ and $\epsilon > 0$. However, this is not quite what we want because of the term $\epsilon|\eta|$. The ϵ term appears as a consequence of the fact that the support of the cutoff function is slightly larger than the support of the ultradistribution itself, as we have discussed before. Furthermore, the constant in (4.7) depends on ϵ as a consequence of the estimate discussed in Example 3.6. Because of this, we cannot simply let $\epsilon \rightarrow 0^+$, and we are instead forced to employ some heavier machinery of complex analysis, namely the Phragmén-Lindelöf Theorem. We note that the following construction is due to Komatsu [Kom77, Theorem 1.1].

Let $w, v \in \mathbb{R}^d$ with $|v| = 1$, and $z \in \overline{\mathbb{C}}_+$. Then every $\zeta = \xi + i\eta \in \mathbb{C}^d$ can be decomposed into a sum of the form $w + vz$. In fact, this is trivial when $\eta = 0$, and otherwise we can write

$$\zeta = (\xi - \eta) + \frac{\eta}{|\eta|}(|\eta| + i|\eta|). \quad (4.8)$$

This decomposition is useful to us since it allows us to use the Phragmén-Lindelöf Theorem for the upper-half plane. If $z = x + iy \in \overline{\mathbb{C}}_+$, we can restate the estimate (4.7) as

$$|\Psi(w + vz)| \leq C_{\lambda, \epsilon} \exp \left(\lambda |w + vz|^{\frac{1}{s}} + yH_K(v) + \epsilon|y| \right), \quad (4.9)$$

for every $\lambda > 0$, and where $H_K(\text{Im}(w + vz)) = yH_K(v)$ since $y \geq 0$ on $\overline{\mathbb{C}}_+$.

Now the idea is to multiply Ψ by certain functions so that the estimate above contains only the ϵ term. More precisely, we should find functions f_1 and f_2 that are holomorphic on $\mathbb{C}_+$, continuous on $\overline{\mathbb{C}}_+$, and such that

$$|f_1(z)f_2(z)\Psi(w + vz)| \leq C_{\lambda, \epsilon} e^{\epsilon|y|},$$

where f_1 takes out $\exp \left(\lambda |w + vz|^{\frac{1}{s}} \right)$ and f_2 takes out $\exp(yH_K(v))$. The Phragmén-Lindelöf Theorem will then allow us to get rid of the ϵ term, so that

$$|\Psi(w + vz)| \leq \frac{C_\lambda}{|f_1(z)f_2(z)|}.$$

To obtain the wanted estimate (4.6), we should further require the functions $\frac{1}{f_1}$ and $\frac{1}{f_2}$ to be bounded by $\exp \left(\lambda |w + vz|^{\frac{1}{s}} \right)$ (with possibly some constant in front of λ) and $\exp(yH_K(v))$ respectively. The choice for f_2 is easy, since we may simply put $f_2(z) = \exp(izH_K(v))$, which has the required properties. As for f_1 , the function $\frac{1}{\Pi}$ from Lemma 4.18 seems to be a good candidate, but the problem is that Π only accepts arguments from $\mathbb{C}$, whereas $w + vz$ is a d -dimensional complex vector, so that we cannot simply insert it into Π . However, we can find a complex number whose modulus behaves in a similar way as $w + vz$, as shown by the construction that follows.

Let ρv be the orthogonal projection of w onto v , i.e. $\rho = \langle w, v \rangle$, and put $q = w - \rho v$. Consider the complex number $z + \rho + i|q|$. Then a direct computation (which we

include in Appendix B) shows that

$$|w + vz| \leq |z + \rho + i|q|| \lesssim |w + vz|, \quad z \in \overline{\mathbb{C}}_+. \quad (4.10)$$

Furthermore, since $z + \rho + i|q| \in \overline{\mathbb{C}}_+$ whenever $z \in \overline{\mathbb{C}}_+$, the function $f_2(z) = \frac{1}{\Pi(z + \rho + i|q|)}$ has the properties that we are after. Let us leave the rest of the details for the proof that follows.

Proof of Theorem 4.17. First suppose that $u \in \mathcal{E}'_s$ with $\text{supp}(u) \subseteq K$, and let Ψ be its Fourier-Laplace transform, i.e.

$$\Psi(\zeta) = u \left(e^{-i\langle \cdot, \zeta \rangle} \right). \quad (4.11)$$

If we consider $\Psi(\zeta) = \Psi(\xi, \eta)$ as a map from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{C}$, Proposition 4.13 implies that Ψ is differentiable on $\mathbb{R}^d \times \mathbb{R}^d$, and in particular, it shows that the Cauchy-Riemann operator commutes with u , i.e.

$$\frac{\partial \Psi}{\partial \bar{\zeta}_j}(\zeta) = \frac{\partial}{\partial \bar{\zeta}_j} u \left(e^{-i\langle \cdot, \zeta \rangle} \right) = u \left(\frac{\partial}{\partial \bar{\zeta}_j} e^{-i\langle \cdot, \zeta \rangle} \right) = 0,$$

for every $1 \leq j \leq d$ and $\zeta \in \mathbb{C}^d$, and where the final equality follows since the exponential is entire. Hence, Proposition 2.15 shows that Ψ is entire.

We continue by showing that Ψ satisfies the growth estimate (4.6). Let $\chi_\epsilon \in \mathcal{G}_c^s$ be a cutoff function with $\text{supp}(\chi_\epsilon) = K_\epsilon = \{x + y : x \in K, |y| \leq \epsilon\}$ and such that $\chi_\epsilon = 1$ on a neighbourhood of K . Recall that we have constructed such a function in Example 3.6. Then it follows from Proposition 4.9 that

$$|\Psi(\zeta)| \leq C_\tau \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K_\epsilon}} \frac{|\partial^\alpha (e^{-i\langle x, \zeta \rangle} \chi_\epsilon(x))|}{\tau^{|\alpha|} \alpha!^s},$$

for every $\tau > 0$ and $\epsilon > 0$. The Leibniz formula gives

$$\begin{aligned} \left| \partial^\alpha \left(e^{-i\langle x, \zeta \rangle} \chi_\epsilon(x) \right) \right| &= \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta e^{-i\langle x, \zeta \rangle} \partial^{\alpha-\beta} \chi_\epsilon(x) \right| \\ &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| \zeta^\beta e^{-i\langle x, \zeta \rangle} \right| \left| \partial^{\alpha-\beta} \chi_\epsilon(x) \right| \\ &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\zeta|^{|\beta|} \left| \partial^{\alpha-\beta} \chi_\epsilon(x) \right| e^{\langle x, \eta \rangle}. \end{aligned}$$

As we have seen in Example 3.6, we can find a constant $\tau_\epsilon > 0$ such that

$$\begin{aligned} \sup_{x \in K_\epsilon} \left| \partial^\alpha \left(e^{-i\langle x, \zeta \rangle} \chi_\epsilon(x) \right) \right| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\zeta|^{|\beta|} \sup_{x \in K_\epsilon} \left\{ \left| \partial^{\alpha-\beta} \chi_\epsilon(x) \right| e^{\langle x, \eta \rangle} \right\} \\ &\lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\zeta|^{|\beta|} \tau_\epsilon^{|\alpha|-|\beta|} (\alpha - \beta)!^s e^{H_{K_\epsilon}(\eta)} \\ &\leq (2\tau_\epsilon)^{|\alpha|} \sup_{\beta \leq \alpha} \left\{ |\zeta|^{|\beta|} \tau_\epsilon^{-|\beta|} (\alpha - \beta)!^s \right\} e^{H_{K_\epsilon}(\eta)}, \end{aligned}$$

where in the final step we use the fact that $\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|}$. Since $\frac{(\alpha-\beta)!}{\alpha!} \leq \frac{1}{\beta!}$, we obtain for arbitrary $\lambda > 0$ that

$$\begin{aligned} \sup_{x \in K_\epsilon} \frac{|\partial^\alpha (e^{-i\langle x, \zeta \rangle} \chi_\epsilon(x))|}{\tau^{|\alpha|} \alpha!^s} &\lesssim \left(\frac{2\tau_\epsilon}{\tau} \right)^{|\alpha|} \sup_{\beta \leq \alpha} \left\{ \tau_\epsilon^{-|\beta|} |\zeta|^{|\beta|} \frac{(\alpha - \beta)!^s}{\alpha!^s} \right\} e^{H_{K_\epsilon}(\eta)} \\ &\leq \left(\frac{2\tau_\epsilon}{\tau} \right)^{|\alpha|} \sup_{\beta \leq \alpha} \left\{ (\tau_\epsilon \lambda^s)^{-|\beta|} \frac{(\lambda |\zeta|^{\frac{1}{s}})^{s|\beta|}}{\beta!^s} \right\} e^{H_{K_\epsilon}(\eta)} \\ &\leq \left(\frac{2\tau_\epsilon}{\tau} \right)^{|\alpha|} \sup_{\beta \leq \alpha} \left\{ \left(\frac{(ds)^s}{\tau_\epsilon \lambda^s} \right)^{|\beta|} \frac{\left(\frac{\lambda}{s} |\zeta|^{\frac{1}{s}} \right)^{s|\beta|}}{|\beta|!^s} \right\} e^{H_{K_\epsilon}(\eta)} \\ &\leq \left(\frac{2\tau_\epsilon M_{\epsilon, \lambda}}{\tau} \right)^{|\alpha|} \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_{K_\epsilon}(\eta) \right), \end{aligned}$$

with a suitably chosen constant $M_{\epsilon, \lambda}$, and where in the final step we employ the inequality $\frac{t^{sn}}{n!^s} \leq e^{st}$. But τ may be chosen arbitrarily, so that putting $\tau = 2\tau_\epsilon M_{\epsilon, \lambda}$ gives

$$\begin{aligned} |\Psi(\zeta)| &\lesssim C_\tau \sup_{\alpha \in \mathbb{N}^d} \left(\frac{2\tau_\epsilon M_{\epsilon, \lambda}}{\tau} \right)^{|\alpha|} \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_{K_\epsilon}(\eta) \right) \\ &= C_{\lambda, \epsilon} \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_{K_\epsilon}(\eta) \right) \\ &\leq C_{\lambda, \epsilon} \exp \left(\lambda |\zeta|^{\frac{1}{s}} + H_K(\eta) + \epsilon |\eta| \right), \end{aligned} \tag{4.12}$$

for every $\lambda > 0$ and $\epsilon > 0$.

In view of the discussion before the proof, let $w, v \in \mathbb{R}^d$ with $|v| = 1$, and $z = x + iy \in \overline{\mathbb{C}}_+$. Also let $\rho = \langle w, v \rangle$, and put $q = w - \rho v$. Fix an arbitrary $\lambda > 0$ and define Π as in Lemma 4.18 with ζ replaced by $z + \rho + i|q|$, i.e.

$$\Pi(z) = \prod_{j=1}^{\infty} \left(1 - \frac{i\lambda^s(z + \rho + i|q|)}{j^s} \right).$$

Then part (b) of Lemma 4.18 shows that $\frac{1}{\Pi} \in \mathcal{H}(\mathbb{C}_+) \cap \mathcal{C}(\overline{\mathbb{C}}_+)$, since $z + \rho + i|q| \in \overline{\mathbb{C}}_+$ whenever $z \in \overline{\mathbb{C}}_+$. Furthermore, part (c) of Lemma 4.18 combined with (4.10) shows that there exists a constant $L > 0$ such that

$$\exp\left(\lambda|w + vz|^{\frac{1}{s}}\right) \lesssim |\Pi(z)| \lesssim \exp\left(L\lambda|w + vz|^{\frac{1}{s}}\right), \quad (4.13)$$

for every $z \in \overline{\mathbb{C}}_+$.

Now define the function $\Theta : \overline{\mathbb{C}}_+ \rightarrow \mathbb{C}$ by

$$\Theta(z) = \frac{\Psi(w + vz)}{\Pi(z)} e^{izH_K(v)}.$$

Then $\Theta \in \mathcal{H}(\mathbb{C}_+) \cap \mathcal{C}(\overline{\mathbb{C}}_+)$ as each of the factors in Θ belongs to $\mathcal{H}(\mathbb{C}_+) \cap \mathcal{C}(\overline{\mathbb{C}}_+)$. Furthermore, it follows from (4.9) and (4.13) that

$$|\Theta(z)| \lesssim C_{\lambda, \epsilon} e^{\epsilon|y|} \leq C_{\lambda, \epsilon} e^{\epsilon|z|}, \quad z = x + iy \in \mathbb{C}_+,$$

for every $\epsilon > 0$, and

$$|\Theta(x)| \lesssim C_{\lambda, \epsilon} = C_\lambda, \quad x \in \mathbb{R},$$

where in the last equality we may take ϵ to be any fixed positive number. Relying upon the Phragmén-Lindelöf Theorem, we conclude that

$$|\Theta(z)| \lesssim C_\lambda, \quad z \in \overline{\mathbb{C}}_+.$$

A combination of this estimate and (4.13) gives

$$|\Psi(w + vz)| \lesssim C_\lambda |\Pi(z)| \left| e^{-izH_K(v)} \right| \lesssim C_\lambda \exp\left(L\lambda|w + vz|^{\frac{1}{s}} + yH_K(v)\right). \quad (4.14)$$

But since λ was arbitrary, we may replace $L\lambda$ in this estimate simply by λ . In view of the discussion in (4.8), the estimate (4.14) is equivalent to

$$|\Psi(\zeta)| \lesssim C_\lambda \exp\left(\lambda|\zeta|^{\frac{1}{s}} + H_K(\eta)\right),$$

for every $\zeta \in \mathbb{C}^d$ and $\lambda > 0$, finishing the proof of necessity.

To prove sufficiency, suppose that Ψ is an entire function that satisfies the growth estimate 4.6. Define for $\phi \in \mathcal{G}_c^s$ the functional

$$u(\phi) = \int_{\mathbb{R}^d} \Psi(-\xi) \widehat{\phi}(\xi) d\mu_d(\xi). \quad (4.15)$$

The integral converges, since by the growth estimate (4.6) combined with the Paley-Wiener Theorem for $\mathcal{G}_c^s$, we get

$$\int_{\mathbb{R}^d} \left| \Psi(-\xi) \widehat{\phi}(\xi) \right| d\mu_d(\xi) \lesssim \int_{\mathbb{R}^d} \exp\left((\lambda - \lambda') |\xi|^{\frac{1}{s}}\right) d\mu_d(\xi), \quad (4.16)$$

for every $\lambda > 0$ and some $\lambda' > 0$. In particular, if we choose λ strictly smaller than λ' , we see that $\Psi(-\xi) \widehat{\phi}(\xi) \in \mathcal{L}^1$, and u is therefore well-defined on $\mathcal{G}_c^s$.

The linearity of u follows immediately as a consequence of the linearity of the Fourier transform. For continuity, let $\phi \in \mathcal{G}_c^s$ with $\text{supp}(\phi) \subseteq K_0$. By replacing λ with $\frac{\lambda}{4}$ in estimate (4.6), we obtain

$$\begin{aligned} |u(\phi)| &\leq C_\lambda \int_{\mathbb{R}^d} \exp\left(\frac{\lambda}{4}|\xi|^{\frac{1}{s}}\right) |\widehat{\phi}(\xi)| d\mu_d(\xi) \\ &\leq C_\lambda \sup_{\xi \in \mathbb{R}^d} \left\{ \exp\left(\frac{\lambda}{2}|\xi|^{\frac{1}{s}}\right) |\widehat{\phi}(\xi)| \right\} \int_{\mathbb{R}^d} \exp\left(-\frac{\lambda}{4}|\xi|^{\frac{1}{s}}\right) d\mu_d(\xi) \\ &\lesssim C_\lambda \sup_{\xi \in \mathbb{R}^d} \left\{ \sum_{k=0}^{\infty} \frac{\lambda^k |\xi|^{\frac{k}{s}}}{2^k k!} |\widehat{\phi}(\xi)| \right\} \lesssim C_\lambda \sup_{\substack{k \in \mathbb{N} \\ \xi \in \mathbb{R}^d}} \left\{ \frac{\lambda^k}{k!} |\xi|^{\frac{k}{s}} |\widehat{\phi}(\xi)| \right\}, \end{aligned} \quad (4.17)$$

for every $\lambda > 0$. Now let $\tau > 0$ be arbitrary. Lemma 3.9 implies the existence of a constant $M > 0$ such that

$$\sup_{k \in \mathbb{N}} \left\{ \frac{\lambda^k}{k!} |\xi|^{\frac{k}{s}} \right\} \lesssim \sup_{\alpha \in \mathbb{N}^d} \left\{ \left(\frac{M\lambda}{|\alpha|} \right)^{s|\alpha|} |\xi^\alpha| \right\} \leq \sup_{\alpha \in \mathbb{N}^d} \frac{|\xi^\alpha|}{\tau^{|\alpha|} \alpha!^s}, \quad (4.18)$$

where in the final step we put $\lambda = (\tau^{\frac{1}{s}} M)^{-1}$. Furthermore, since ϕ is supported in K_0 , integration by parts as in (3.7) gives

$$\sup_{\xi \in \mathbb{R}^d} |\xi^\alpha \widehat{\phi}(\xi)| = \sup_{\xi \in \mathbb{R}^d} |(\widehat{\partial^\alpha \phi})(\xi)| \leq \int_{K_0} |\partial^\alpha \phi(x)| d\mu_d(x) \lesssim \sup_{x \in K_0} |\partial^\alpha \phi(x)|. \quad (4.19)$$

Thus, (4.17) combined with (4.18) and (4.19) gives

$$|u(\phi)| \lesssim C_\lambda \sup_{\substack{\alpha \in \mathbb{N}^d \\ \xi \in \mathbb{R}^d}} \frac{|\xi^\alpha \widehat{\phi}(\xi)|}{\tau^{|\alpha|} \alpha!^s} \lesssim C_\lambda \sup_{\substack{\alpha \in \mathbb{N}^d \\ x \in K_0}} \frac{|\partial^\alpha \phi(x)|}{\tau^{|\alpha|} \alpha!^s},$$

for every $\tau > 0$ and $\phi \in \mathcal{G}_c^s$ with $\text{supp}(\phi) \subseteq K_0$, so that $u \in \mathcal{D}'_s$ by Proposition 4.3.

To show that $\text{supp}(u) \subseteq K$, we will show that the integral in 4.15 is invariant under the variable change $\xi \mapsto \xi + i\eta$, and then a particular choice of η will give us the wanted result in a similar way as in the proof of the Paley-Wiener Theorem for $\mathcal{G}_c^s$. Let us once more introduce the notation

$$\zeta'(z) = (\zeta_1, \dots, z, \dots, \zeta_d),$$

where z replaces the j -th coordinate of ζ and the other coordinates are fixed as before. Let Γ_ρ be the same contour as in the proof of Theorem 3.8. As we have already seen, it suffices to show that the integrals over the vertical edges vanish as $\rho \rightarrow \infty$. Indeed,

if K_0 is the convex hull of $\text{supp}(\phi)$, then the Paley-Wiener Theorem for $\mathcal{G}_c^s$ implies the existence of a $\lambda' > 0$ such that

$$|\widehat{\phi}(\zeta'(z))| \lesssim \exp\left(-\lambda'|\zeta'(z)|^{\frac{1}{s}} + H_{K_0}(\text{Im } \zeta'(z))\right),$$

and by choosing $\lambda < \lambda'$, we obtain

$$\begin{aligned} |\Psi(-\zeta'(z)) \widehat{\phi}(\zeta'(z))| &\leq \exp\left((\lambda - \lambda')|\zeta'(z)|^{\frac{1}{s}} + H_K(-\text{Im } \zeta'(z)) + H_{K_0}(\text{Im } \zeta'(z))\right) \\ &\lesssim |\zeta'(z)|^{-\frac{n}{s}} \exp\left(\sup_{t \in K} \langle t, -\text{Im } \zeta'(z) \rangle + \sup_{t \in K_0} \langle t, \text{Im } \zeta'(z) \rangle\right) \\ &\lesssim |\text{Re } z|^{-\frac{n}{s}} \exp\left(\sup_{t \in K} |t| |\text{Im } \zeta'(z)| + \sup_{t \in K_0} |t| |\text{Im } \zeta'(z)|\right), \quad (4.20) \end{aligned}$$

where in the second step we employ the estimate $e^{-t} \leq n!t^{-n}$, and in the final step we use the Cauchy-Schwartz inequality. But $|\text{Im } \zeta'(z)| \leq |\eta|$ on the vertical edges, and since η is fixed, it follows that the exponential in (4.20) is fixed as well. Furthermore, $|\text{Re } z| = \rho$ on the vertical edges, and the integrals over the vertical edges therefore converge to 0 as $\rho \rightarrow \infty$, as we see from (4.20). Thus,

$$u(\phi) = \int_{\mathbb{R}^d} \Psi(-\zeta) \widehat{\phi}(\zeta) d\mu_d(\xi),$$

for all $\zeta = \xi + i\eta \in \mathbb{C}^d$.

Now suppose that $\phi \in \mathcal{G}_c^s$ with $\text{supp}(\phi) \subseteq K_0$ where K_0 is a compact convex set such that $K \cap K_0 = \emptyset$. Utilizing the Paley-Wiener Theorem for $\mathcal{G}_c^s$ one more time, we obtain

$$\begin{aligned} |u(\phi)| &\leq \int_{\mathbb{R}^d} \left| \Psi(-\xi - i\eta) \widehat{\phi}(\xi + i\eta) \right| d\mu_d(\xi) \\ &\leq \int_{\mathbb{R}^d} \exp\left((\lambda - \lambda')|\zeta|^{\frac{1}{s}} + H_K(-\eta) + H_{K_0}(\eta)\right) d\mu_d(\xi) \\ &= \exp\left(\sup_{t \in K_0} \langle t, \eta \rangle - \inf_{t \in K} \langle t, \eta \rangle\right) \int_{\mathbb{R}^d} \exp\left((\lambda - \lambda')|\zeta|^{\frac{1}{s}}\right) d\mu_d(\xi), \end{aligned}$$

where the integral converges as we have seen in (4.16). Since K and K_0 are a pair of disjoint compact convex sets, Theorem 2.4 shows that we can find a unit vector $v \in \mathbb{R}^d$ such that $\sup_{t \in K_0} \langle t, v \rangle - \inf_{t \in K} \langle t, v \rangle < 0$. Thus, if we choose $\eta = \gamma v$ for $\gamma \in \mathbb{R}_+$, we get

$$|u(\phi)| \lesssim \exp\left(\sup_{t \in K_0} \langle t, \gamma v \rangle - \inf_{t \in K} \langle t, \gamma v \rangle\right) = \exp\left(\gamma \left(\sup_{t \in K_0} \langle t, v \rangle - \inf_{t \in K} \langle t, v \rangle\right)\right).$$

Letting $\gamma \rightarrow \infty$ therefore shows that $u(\phi) = 0$ for every $\phi \in \mathcal{G}_c^s$ supported in a compact convex set disjoint from K .

Now recall that every $\varphi \in \mathcal{G}_c^s$ supported in an arbitrary compact set disjoint from K can be decomposed into a finite sum of the form

$$\varphi = \sum_{j=1}^n \phi_j,$$

where $\phi_j \in \mathcal{G}_c^s$ are supported in convex compact sets also disjoint from K , as we have motivated in Example 3.7. It follows by the linearity of u that

$$u(\varphi) = \sum_{j=1}^n u(\phi_j) = 0,$$

for every $\varphi \in \mathcal{G}_c^s$ with support in a compact set disjoint from K , so that $\text{supp}(u) \subseteq K$, and thus $u \in \mathcal{E}'_s$.

Finally, we must show that Ψ is the Fourier-Laplace transform of u . Let $\{\psi_\epsilon\}_{\epsilon>0}$ be the approximate identity from Example 3.5, and define temporarily $u_\epsilon = u * \psi_\epsilon$. For every $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathcal{F}(\psi_\epsilon(x - \cdot))(\xi) &= \int_{\mathbb{R}^d} \psi_\epsilon(x - y) e^{-i\langle y, \xi \rangle} d\mu_d(y) \\ &= e^{-i\langle x, \xi \rangle} \int_{\mathbb{R}^d} \psi_\epsilon(y) e^{-i\langle y, -\xi \rangle} d\mu_d(y) = \widehat{\psi}_\epsilon(-\xi) e^{-i\langle x, \xi \rangle}, \end{aligned}$$

which gives

$$u_\epsilon(x) = u(\psi_\epsilon(x - \cdot)) = \int_{\mathbb{R}^d} \Psi(-\xi) \widehat{\psi}_\epsilon(-\xi) e^{-i\langle x, \xi \rangle} d\mu_d(\xi) = \int_{\mathbb{R}^d} \Psi(\xi) \widehat{\psi}_\epsilon(\xi) e^{i\langle x, \xi \rangle} d\mu_d(\xi).$$

But since $u_\epsilon \in \mathcal{G}_c^s \subseteq \mathcal{L}^1$ by Proposition 4.14 and $\Psi \widehat{\psi}_\epsilon \in \mathcal{L}^1$ as we have seen in (4.16), the Inversion Theorem may be applied to obtain

$$\widehat{u}_\epsilon(\xi) = \Psi(\xi) \widehat{\psi}_\epsilon(\xi), \quad \xi \in \mathbb{R}^d. \quad (4.21)$$

Letting $\epsilon \rightarrow 0^+$, we obtain by part (b) of Proposition 4.14 that

$$\widehat{u}_\epsilon(\xi) = u_\epsilon \left(e^{-i\langle \cdot, \xi \rangle} \right) \rightarrow u \left(e^{-i\langle \cdot, \xi \rangle} \right) = \widehat{u}(\xi), \quad \xi \in \mathbb{R}^d,$$

so that the left hand side of (4.21) converges pointwise to the Fourier transform of u . Furthermore, $\widehat{\psi}_\epsilon$ converges pointwise to the identity function as we have seen in Example 3.5, so (4.21) shows that $\widehat{u}(\xi) = \Psi(\xi)$ for every $\xi \in \mathbb{R}^d$. But then Ψ and the Fourier-Laplace transform of u are entire functions that coincide on $\mathbb{R}^d$, so that they must coincide on the whole $\mathbb{C}^d$ by Proposition 2.24. *Q.E.D.*

We finish our analysis with another example.

4.19 Example. Let $\Psi : \mathbb{C} \rightarrow \mathbb{C}$ be the function

$$\Psi(\zeta) = \prod_{j=1}^{\infty} \left(1 + \frac{\zeta}{j^{s+1}}\right) \prod_{j=1}^{\infty} \cos\left(\frac{\zeta}{j^s}\right).$$

Let us show that Ψ is the Fourier-Laplace transform of some ultradistribution $u \in \mathcal{E}'_s$ using the Paley-Wiener Theorem.

To show that Ψ is entire, we will show that each of the infinite products in Ψ forms an entire function. For the first product, we obtain directly that

$$\sum_{j=1}^{\infty} \frac{|\zeta|}{j^{s+1}} \leq |\zeta| \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \lesssim |\zeta|,$$

which shows that the sum converges uniformly on compact subsets of $\mathbb{C}$ and the infinite product is therefore entire by Proposition 2.21. For the second product, we find by Taylor expanding the cosine that

$$|1 - \cos(\zeta)| = \left| \sum_{k=1}^{\infty} \frac{(-1)^k \zeta^{2k}}{(2k)!} \right| \leq |\zeta| \sum_{k=1}^{\infty} \frac{|\zeta|^{2k-1}}{(2k)!} \leq |\zeta| \sum_{k=1}^{\infty} \frac{1}{(2k)!} \lesssim |\zeta|,$$

given that $|\zeta| \leq 1$. It follows in the same way as in Example 3.10 that

$$\sum_{j=1}^{\infty} |1 - \cos(j^{-s}\zeta)|$$

converges uniformly on compact subsets of $\mathbb{C}$. Thus, each of the factors in Ψ is entire, so that Ψ is entire as well.

We continue by showing that Ψ satisfies the required growth condition. Note that

$$1 + \frac{|\zeta|}{j^{s+1}} \leq C_{\lambda} \left(1 + \frac{\lambda|\zeta|}{j^{s+1}}\right),$$

for every $\lambda > 0$ and some constant $C_{\lambda} > 0$. Indeed, if $\lambda \leq 1$, then the estimate holds with $C_{\lambda} = \frac{1}{\lambda}$, and if $\lambda > 1$, then the inequality is immediate with $C_{\lambda} = 1$. Now if $|\zeta| \leq 1$, we let m be an integer such that $s-1 < m \leq s$, and if $|\zeta| > 1$, we let m be an integer such that $s \leq m < s+1$. Then since $1 + \frac{t^m}{m!} \leq e^t$, we have

$$\begin{aligned} 1 + \frac{\lambda|\zeta|}{j^{s+1}} &= 1 + \frac{j^{-(s+1)} m! \lambda |\zeta|}{m!} = 1 + \frac{\left(j^{-(s+1)} m! \lambda\right)^{\frac{1}{m}} |\zeta|^{\frac{1}{m}}}{m!}^m \\ &\leq \exp \left(\left(j^{-(s+1)} m! \lambda\right)^{\frac{1}{m}} |\zeta|^{\frac{1}{m}} \right) \leq \exp \left(j^{-\frac{s+1}{m}} (m! \lambda)^{\frac{1}{m}} |\zeta|^{\frac{1}{s}} \right), \end{aligned}$$

for all $\lambda > 0$ and $\zeta \in \mathbb{C}$. Hence,

$$\begin{aligned} \prod_{j=1}^{\infty} \left(1 + \frac{\lambda |\zeta|}{j^{s+1}} \right) &\leq C_{\lambda} \prod_{j=1}^{\infty} \exp \left(j^{-\frac{s+1}{m}} (m! \lambda)^{\frac{1}{m}} |\zeta|^{\frac{1}{s}} \right) \\ &= C_{\lambda} \exp \left((m! \lambda)^{\frac{1}{m}} |\zeta|^{\frac{1}{s}} \sum_{j=1}^{\infty} j^{-\frac{s+1}{m}} \right) \\ &= C_{\lambda} \exp \left(L (m! \lambda)^{\frac{1}{m}} |\zeta|^{\frac{1}{s}} \right), \end{aligned}$$

where the sum over j converges to some constant $L > 0$ since $\frac{s+1}{m} > 1$ by our choice of m . But since this estimate holds for every $\lambda > 0$, and since L and m are constants depending only on s , we may replace $L (m! \lambda)^{\frac{1}{m}}$ simply by λ to obtain

$$\left| \prod_{j=1}^{\infty} \left(1 + \frac{\zeta}{j^{s+1}} \right) \right| \leq C_{\lambda} \exp \left(\lambda |\zeta|^{\frac{1}{s}} \right),$$

for every $\lambda > 0$ and $\zeta \in \mathbb{C}^d$. Furthermore, we have

$$|\cos \zeta| = \left| \frac{e^{i\zeta} + e^{-i\zeta}}{2} \right| \leq \frac{|e^{-\eta} e^{i\xi}| + |e^{\eta} e^{-i\xi}|}{2} \leq e^{|\eta|},$$

which implies for every $j \geq 1$ the estimate

$$|\cos(j^{-s} \zeta)| \leq \exp(j^{-s} |\eta|), \quad \zeta \in \mathbb{C}.$$

Hence,

$$\prod_{j=1}^{\infty} |\cos(j^{-s} \zeta)| \leq \prod_{j=1}^{\infty} \exp(j^{-s} |\eta|) = \exp \left(\sum_{j=1}^{\infty} j^{-s} |\eta| \right) = e^{R_s |\eta|},$$

with $R_s = \sum_{j=1}^{\infty} j^{-s}$. It follows that

$$|\Psi(\zeta)| \leq C_{\lambda} \exp \left(\lambda |\zeta|^{\frac{1}{s}} + R_s |\eta| \right),$$

for every $\lambda > 0$ and $\zeta \in \mathbb{C}$. Utilizing the Paley-Wiener Theorem for $\mathcal{E}'_s$, we see that Ψ is the Fourier-Laplace transform of some ultradistribution $u \in \mathcal{E}'_s$ with $\text{supp}(u) \subseteq [-R_s, R_s]$. In particular, the proof of the theorem reveals that u (as an element of $\mathcal{D}'_s$) is given by

$$u(\phi) = \int_{\mathbb{R}} \Psi(-\xi) \widehat{\phi}(\xi) d\mu_1(\xi) = \int_{\mathbb{R}} \widehat{\phi}(\xi) \prod_{j=1}^{\infty} \left(1 - \frac{\xi}{j^{s+1}} \right) \prod_{j=1}^{\infty} \cos \left(-\frac{\xi}{j^s} \right) d\mu_1(\xi).$$

△

5. Discussion

5.1. Summary

The Gevrey classes $\mathcal{G}^s$ were introduced as a means of classification of the regularity of functions, i.e. in some sense measuring how close they are to being real-analytic. We have discussed some properties of this class, and shown how one can modify some useful constructions from the classical $\mathcal{C}_c^\infty$ space to obtain their Gevrey versions.

In particular, we have shown how one can sharpen the classical Paley-Wiener Theorem for $\mathcal{C}_c^\infty$ to obtain a similar result for $\mathcal{G}_c^s$. The main difference between the two is that polynomial decay of the Fourier-Laplace transform is replaced by an exponential decay with some dependence on the parameter s . The proof that we presented was largely based on the proofs given in [Hör90, Theorem 7.3.1] and [Rud74, Theorem 7.22], with some minor tweaks which were needed specifically for the Gevrey class.

We then presented a non-trivial example of a compactly supported Gevrey function, and we showed that this function truly belongs to $\mathcal{G}_c^s$ with the help of the Paley-Wiener Theorem for $\mathcal{G}_c^s$.

Following the ideas from classical distribution theory, we then proceeded to analyze the spaces $\mathcal{D}'_s$ and $\mathcal{E}'_s$ of Gevrey ultradistributions, defined as continuous linear functionals on the spaces $\mathcal{G}_c^s$ and $\mathcal{G}^s$ respectively. We have shown how certain notions and results from classical distribution theory can be extended to ultradistributions.

The main result that we have shown is the Paley-Wiener Theorem for $\mathcal{E}'_s$, which is analogous to its classical counterpart for $\mathcal{E}'$, the only difference being that we require exponential growth with some dependence on the parameter s instead of polynomial growth of the Fourier-Laplace transform. Our proof was based mostly on [Hör90, Theorem 7.3.1], [Rud74, Theorem 7.23], and [Kom77, Theorem 1.1].

One of the difficulties that we encountered while trying to modify the theorems from the references above was that in the classical case, one can bound a distribution by a finite sum over multi-indices α , as shown by Proposition 2.27, whereas an ultradistribution is bounded by a supremum over α , as we have seen in Proposition 4.3. This fact made it somewhat more complicated to obtain the wanted estimate (4.6) of the Fourier-Laplace transform. Indeed, the estimate that we obtained by direct estimation depended on an undesired parameter ϵ . However, we succeeded to remove this dependence after a somewhat complicated process involving the Phragmén-Lindelöf Theorem. The motivation for this process and some other specific details regarding

the proof were taken from [Kom77, Theorem 1.1].

Once we completed the proof, we proceeded to present a non-trivial example of a compactly supported Gevrey ultradistribution, and thus showed how one can apply the ultradistributional version of the Paley-Wiener Theorem.

The results that we have presented are greatly significant in the study of partial differential operators. In particular, the Paley-Wiener Theorems allow us to study the differential operators on the Fourier(-Laplace) transform side, while still retaining information regarding the Gevrey regularity of the actual solutions of the operators. For more details on the analysis of partial differential operators on Gevrey spaces, a standard reference is [Rod93].

5.2. Future work

To generalise the concepts and the results presented in this work, one can consider a sequence $\{M_p\}_{p=1}^\infty$ of positive numbers, impose certain conditions on it, and then replace the Gevrey estimate by

$$\sup_{x \in K} |\partial^\alpha f(x)| \lesssim \tau^{|\alpha|} M_{|\alpha|},$$

for every compact set K . In this way one obtains the spaces of the so-called *ultra-differentiable functions*. Depending on whether the estimate above is required to hold for every $\tau > 0$ or only for some $\tau > 0$, one obtains the ultradifferentiable functions of the so-called Beurling type and Roumieu type respectively. For an extensive analysis of these spaces, see e.g. [Kom73]. We note that this is truly a generalisation of the Gevrey class, since if we put $M_p = p!^s$, and consider ultradifferentiable functions of the Roumieu type, we recover precisely the class of Gevrey functions of order s .

For the sequence $\{M_p\}_{p=1}^\infty$ one also defines the associated function as

$$M(t) = \sup_{p \in \mathbb{N}} \log \left(\frac{t^p}{M_p} \right), \quad t \in \mathbb{R}_+.$$

Given that the sequence $\{M_p\}_{p=1}^\infty$ satisfies certain conditions, it is then possible to show that the Paley-Wiener Theorem holds also for spaces of ultradifferentiable functions with compact support, the only difference being that we require the Fourier-Laplace transform to instead satisfy

$$|\Psi(\zeta)| \leq C_\lambda \exp(-M(\lambda|\zeta|) + H_K(\operatorname{Im} \zeta)), \quad \zeta \in \mathbb{C}^d, \quad (5.1)$$

for every $\lambda > 0$ for the Beurling class, and for some $\lambda > 0$ for the Roumieu, see e.g. [Kom73, Theorem 9.1]. We observe that in the case of $M_p = p!^s$, the growth of the associated function can be approximated as $M(|\zeta|) \sim |\zeta|^{\frac{1}{s}}$, so that (5.1) also implies the estimate from the now familiar Paley-Wiener Theorem for $\mathcal{G}_c^s$.

One can also define convergence in the spaces of ultradifferentiable functions analogously as in Definition 4.1 by replacing $\alpha!^s$ by $M_{|\alpha|}$, which then makes it possible to consider more general ultradistributions of Beurling and Roumieu type. The properties of these ultradistributions depend heavily on the conditions that one imposes on the sequence $\{M_p\}_{p=1}^\infty$, but under certain conditions one can show that many of the results presented in this work hold as well. One of them is the Paley-Wiener Theorem, which is again obtained analogously by requiring the Fourier-Laplace transform to instead satisfy

$$|\Psi(\zeta)| \leq C_\lambda \exp(M(\lambda|\zeta|) + H_K(\operatorname{Im} \zeta)), \quad \zeta \in \mathbb{C}^d, \quad (5.2)$$

for some $\lambda > 0$ in the case of Beurling, and every $\lambda > 0$ in the case of Roumieu, see e.g. [Kom77, Theorem 1.1]. As a final note, we observe that the estimate in the Paley-Wiener Theorem for $\mathcal{E}'_s$ is a specific case of (5.2), as motivated before.

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A. Convergence of an integral

We show that the integral

$$\int_0^\infty \log\left(1 + \frac{1}{x^s}\right) dx$$

converges. The estimate $\log(1+t) \leq t$ shows that

$$\int_1^\infty \log\left(1 + \frac{1}{x^s}\right) dx \leq \int_1^\infty \frac{1}{x^s} dx < \infty,$$

where the final integral is convergent since $s > 1$. On the other hand, integration by parts yields

$$\begin{aligned} \int_0^1 \log\left(1 + \frac{1}{x^s}\right) dx &= \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^1 \log\left(1 + \frac{1}{x^s}\right) dx \\ &= \lim_{\epsilon \rightarrow 0^+} \left(\left[x \log\left(1 + \frac{1}{x^s}\right) \right] \Big|_\epsilon^1 + s \int_\epsilon^1 \frac{1}{x^s + 1} dx \right) \\ &= \log 2 + \lim_{\epsilon \rightarrow 0^+} \left(\epsilon \log\left(1 + \frac{1}{\epsilon^s}\right) \right) + s \int_0^1 \frac{1}{x^s + 1} dx < \infty, \end{aligned}$$

where the final step follows since the limit is equal to zero and the final integral is proper. Thus,

$$\int_0^\infty \log\left(1 + \frac{1}{x^s}\right) dx < \infty.$$

B. Estimate of a modulus

Let $w, v \in \mathbb{R}^d$ with $|v| = 1$, and $z \in \overline{\mathbb{C}}_+$. Let ρv be the orthogonal projection of w onto v , i.e. $\rho = \langle w, v \rangle$, and put $q = w - \rho v$. Consider the complex number $z + \rho + i|q|$. Since q and v are orthogonal we obtain for $z = x + iy \in \overline{\mathbb{C}}_+$ that

$$\begin{aligned}
 |w + vz|^2 &= |w|^2 + 2 \operatorname{Re} \langle w, vz \rangle + |vz|^2 \\
 &= |\rho v + q|^2 + 2x \langle \rho v + q, v \rangle + |z|^2 \\
 &= |\rho v|^2 + 2\rho \langle v, q \rangle + |q|^2 + 2x\rho + |z|^2 \\
 &= \rho^2 + 2x\rho + x^2 + y^2 + |q|^2 \\
 &= (\rho + x)^2 + y^2 + |q|^2 \\
 &\leq (\rho + x)^2 + (y + |q|)^2 \\
 &= |(\rho + x) + i(y + |q|)|^2 \\
 &= |z + \rho + i|q||^2,
 \end{aligned}$$

where the inequality follows since $y \geq 0$. On the other hand,

$$\begin{aligned}
 |z + \rho + i|q||^2 &= (x + \rho)^2 + (y + |q|)^2 \\
 &\leq (x + \rho)^2 + 2y^2 + 2|q|^2 \\
 &\leq 2((x + \rho)^2 + y^2 + |q|^2) \\
 &= 2|w + vz|^2.
 \end{aligned}$$

Hence,

$$|w + vz| \leq |z + \rho + i|q|| \lesssim |w + vz|,$$

for every $z \in \overline{\mathbb{C}}_+$.



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